MECHANICS MAP

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Mechanics Map

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About the Authors

Project Lead: Dr. Jacob Moore

Dr. Moore is an Associate Professor of Engineering at Penn State Mont Alto. His research interests include open educational resources in engineering, concept maps in education, student assessment, and additive manufacturing technologies. As the project lead, Dr. Moore oversees all development and evaluation activities and is currently the primary content developer.

Current Content Developers

Dr. Moore has been assisted by Dr. Majid Chatsaz at Penn State Scranton, Dr. Agnes D'entremont at The University of British Colombia, Joan Kowalski at Penn State New Kensington, and Dr. Douglas Miller at Penn State Dubois.

Past Contributors

We would also like to acknowledge past software developers, Nathanael Bice, Lauren Gibboney, Joseph Luke, James McIntyre, John Nein, Tucker Noia, Michel Pascale, Joshua Rush, Shawn Shroyer, and Menelik Young as well as the content experts we have consulted with, Dr. Robert Scott Pierce and Christopher Venters.

Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing***.

About the Book

About the Mechanics Map Tool

The Mechanics Map Digital Textbook Project is an open digital textbook founded on the idea that expert generated concept maps can serve as a powerful advance organizer for textbook content. The overview at the beginning of each chapter consists of a video showing how all the topics in the chapter are linked together by the author. By providing this overview, the author is seeking to help users organize the knowledge they are developing in a way that matches the expert's organization of knowledge.

Theoretical Basis

Advance organizers are high level overviews of more detailed information presented to a learner before detailed instruction in language a novice can understand. This overview will ideally link the new ideas to the learner's prior knowledge. When used with instruction, advance organizers have been shown to have a small but significant positive impact on the understanding and retention of the content.

Concept maps are node-link diagrams that show the major concepts in a content area and how those concepts are linked. Concept maps were originally developed as a way to chart what children did and did not understand, but they were quickly found to be effective as a learning aid. Expert-generated concept maps can serve as particularly powerful advance organizers because of their explicit highlighting of the relationships between concepts.

Project History

Work on the Mechanics Map project began in 2011 with NSF funding to explore the feasibility and usefulness of a content navigation system based on expert generated concept maps. This interactive navigation system replaced a traditional table of contents, and allowed users to navigate the material in a non-linear way, all the while absorbing the expert generated concept map as an advance organizer.

Videos of the original navigation system can be seen below.

The tool was tested in the classroom and was shown to be more effective than a traditional paper textbook in two respects. First, as predicted with the design, the tool encourages users to spend more time attending to an overview of the information, helping students build a skeleton they can fit details into later. Second, the tool encouraged users to step back and review topics from previous sections that were relevant to the topics they were learning. This combination of behaviors in the users leads to greater measures of conceptual understanding, with little to no extra effort on the part of the learner.

Unfortunately the original navigation system, built as a Java Applet, is now inoperable in all major browsers due to the security concerns with Applets. Despite this setback with the software, the content has been significantly expanded to all of engineering statics and engineering dynamics with video lectures, worked examples, and homework problems. Additionally, the video introduction at the beginning of each chapter highlights an expert generated concept map to use as an advance organizer for that chapter's concepts.

Research Publications

Moore, J., Williams, C., North C., Johri, A., Paretti, M. (2015). "Effectiveness of Adaptive Concept Maps for Promoting Conceptual Understanding: Findings from a Design-Based Case Study of a Learner-Centered Tool" Advances in Engineering Education ASEE 4 (4)

Moore, J. Pascale, M., Williams, C. North, C. (2013) *Translating Educational Theory Into Educational Software: A Case Study of the Adaptive Map Project* Proceedings of the 2013 ASEE Annual Conference Atlanta, GA, ASEE.

Moore, J. Pierce, R. S., Williams, C. (2012) *Towards an "Adaptive Concept Map": Creating an Expert-Generated Concept Map of an Engineering Statics Curriculum* Proceedings of the 2012 ASEE Annual Conference San Antonio, TX, ASEE.

CHAPTER OVERVIEW

1: Basics of Newtonian Mechanics

1.0: Video Introduction to Chapter 1

Video introduction to Chapter 1, delivered by Dr. Jacob Moore. Discusses focus of the chapter: the impact of forces and moments on physical bodies. YouTube source: https://www.youtube.com/watch?v=ljkTOjZ6jt8&t=3s.

1.1: Bodies

Bodies in Engineering Mechanics

A **body**, for the purposes of engineering mechanics, is a collection of matter that is analyzed as a single object. This can be something simple like a rubber ball, or it can be something made of many parts such as a car. What can count as a body and what cannot count as a body is dependent on the circumstances of the analysis. In some circumstances in engineering mechanics, it is useful to make certain assumptions about the bodies being analyzed. We will usually need to assume the body is either rigid or deformable, and we will also need to assume that the body is either a particle or an extended body.

Rigid versus Deformable Bodies

Rigid bodies do not deform (stretch, compress, or bend) when subjected to loads, while deformable bodies do deform. In actuality, no physical body is completely rigid, but most bodies deform so little that this deformation has a minimal impact on the analysis. For this reason, we usually assume in the statics and dynamics courses that the bodies discussed are rigid. In the strength of materials course we specifically remove this assumption and examine how bodies deform and eventually fail under loading.

There is no set boundary for determining if a body can be approximated as rigid, but there are two factors to look for that indicate that a rigid body assumption is not appropriate. First, if the body is being significantly stretched, compressed, or bent during the period of analysis, then the body should not be analyzed as a rigid body. Second, if the body has parts that are free to move relative to one another, then the body as a whole should not be analyzed as a rigid body; this is instead a machine, comprised of multiple connected bodies that will each need to be analyzed separately.

Figure 1.1.1: This hammer is a good example of a rigid body for analysis. It deforms little under regular use and does not have any pieces that move relative to one another. Public Domain image, no author listed.

Figure 1.1.2: This car deformed significantly during the crash test. When analyzing the impact, we should not treat the car as a rigid body. Image by Brady Holt CC-BY-3.0.

Figure 1.1.3: This pair of scissors consists of two halves held together with a rivet. Because the two halves can move relative to one another, the pair of scissors as whole should not be treated as a rigid body. Image by ZooFari CC-BY-SA 3.0.

Particles versus Extended Bodies

Particles are bodies where all the mass is concentrated at a single point in space. Particle analysis will only have to take into account translational motion and the forces acting on the body, because rotation is not considered for particles. Extended bodies, on the other hand, have mass that is distributed throughout a finite volume. Often in engineering statics, we will take a shortcut and say **rigid bodies** to describe extended bodies that also happen to be rigid. This is because particles, as a single point, cannot deform. Extended body analysis is more complex and also has to take into account moments and rotational motions. In actuality, no bodies are truly particles, but some bodies can be approximated as particles to simplify analysis. Bodies are often assumed to be particles if the rotational motions are negligible when compared to the translational motions, or in systems where there is no moment exerted on the body, such as a concurrent force system.

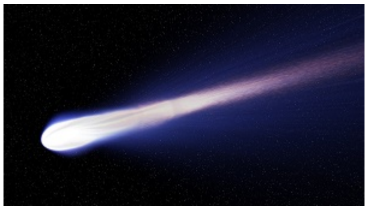


Figure 1.1.4: The rotation of this comet and the moments exerted on the comet are unimportant in modeling its trajectory through space, therefore we would treat it as a particle. Public Domain image by Buddy Nath.

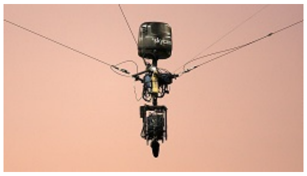


Figure 1.1.5: The gravitational forces and the tension forces on the skycam all act through a single point, making this a concurrent force system that can be analyzed as a particle. Image by Despeaux CC-BY-SA 3.0.

Figure 1.1.6: Rotation and moments will be key to the analysis of the crowbar in this system, therefore the crowbar needs to be analyzed as an extended body. Public Domain image by Pearson Scott Foresman.

Video 1.1.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/-ETzKW31aZI.

1.2: Forces

A force is any influence that causes a body to accelerate. Forces on a body can also cause stress in that body, which can result in the body deforming or breaking. Though forces can come from a variety of sources, there are three distinguishing features to every force. These features are the **magnitude** of the force, the **direction** of the force, and the **point of application** of the force. Forces are often represented as **vectors** (as in the diagram to the right) and each of these features can be determined from a vector representation of the forces on the body.

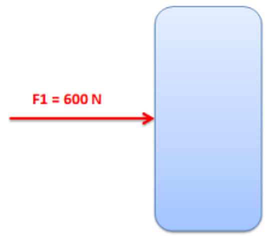


Figure 1.2.1: A basic point force acting on a body.

Magnitude:

The magnitude of a force is the degree to which the force will accelerate the body it is acting on; it is represented by a scalar (a single number). The magnitude can also be thought of as the strength of the force. When forces are represented as vectors, the magnitude of the force is usually explicitly labeled. The length of the vector also often corresponds to the relative magnitude of the vector, with longer vectors indicating larger magnitudes.

The magnitude of force is measured in units of mass times length over time squared. In metric units the most common unit is the Newton (N), where one Newton is one kilogram times one meter over one second squared. This means that a force of one Newton would cause a one-kilogram object to accelerate at a rate of one meter per second squared. In English units, the most common unit is the pound (lb), where one pound is equal to one slug times a foot over a second squared. This means that a one-pound force would cause an object with a mass of one slug to accelerate at a rate of one foot per second squared.

$$Force = \frac{(mass)(distance)}{(time)^2} \tag{1.2.1}$$

$$1\,\text{Newton}\,(N) = \frac{(kg)(m)}{s^2} \tag{1.2.2}$$

$$1\,\text{pound}\,(lb) = \frac{(slug)(ft)}{s^2} \tag{1.2.3}$$

Direction:

In addition to having magnitudes, forces also have directions. As we said before, a force is any influence that causes a body to accelerate. Since acceleration has a specific direction, force also has a specific direction that matches this acceleration. The direction of the force is indicated in diagrams by the direction of the vector representing the force.

Direction has no units, but it is usually given by reporting angles between the vector representing the force and coordinate axes, or by reporting the X, Y, and Z components of the vector. Often times vectors that have the same direction as one of the coordinate axes will not have any angles or components listed. If this is true, it is usually safe to assume that the direction does match the direction of one of the coordinate axes.

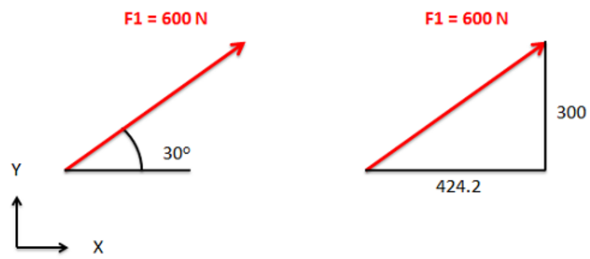


Figure 1.2.2: The magnitude and direction of a vector can be given as a magnitude and an angle of the vector, or by giving magnitude of the vector components in each of the coordinate axes.

Point of Application:

The point, or points, at which a force is applied to a body is important for understanding how the body will react. For particles, there is only a single point for the forces to act on, but for rigid bodies there are an infinite number of possible points of application. Some points of application will lead to the body undergoing simple linear acceleration; some will exert a moment on the body which will cause the body to undergo rotational acceleration as well as linear acceleration.

Depending on the nature of the point of application of a force, there are three general types of forces. These are **point forces**, **surface forces**, and **body forces**. Below is a diagram of a box being pulled by a rope across a frictionless surface. The box has three forces acting on it. The first is the force from the rope. This is a force applied to a single point on the box, and is therefore modeled as point force. Point forces are represented by a single vector. Second is the normal force from the ground that is supporting the box. Because this force is applied evenly to the bottom surface of the box, it is best modeled as a surface force. Surface forces are indicated by a number of vectors drawn side by side with a profile line to indicate the magnitude of the force at any point. The last force is the gravitational force pulling the box downward. Because this force is applied evenly to the entire volume of the box, it is best modeled as a body force. Body forces are sometimes shown as a field of vectors as shown, though they are often not drawn out at all because they end up cluttering the free body diagram.

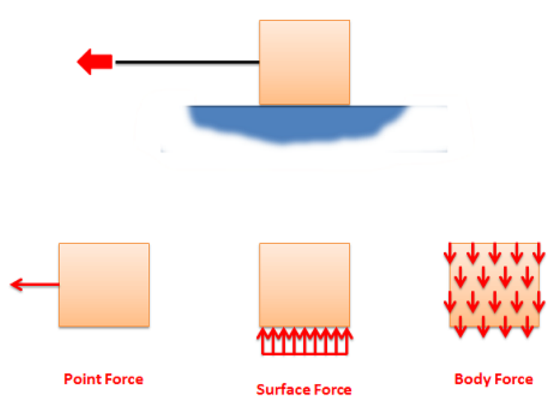


Figure 1.2.3: The box being pulled along a frictionless surface as shown above has three types of forces acting on it. The tension in the cable is best represented by a point force, the normal force supporting the box is best represented by a surface force, and the gravitational force on the box is best represented as a body force.

We will also sometimes talk about **distributed forces**. A distributed force is simply another name for either a surface or a body force.

The exact point or surface that the force is acting on can be drawn as either the head or the tail of the force vector in the free body diagram. Because of the **principle of transmissibility**, both options are known to represent the same physical system.

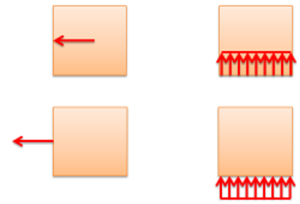


Figure 1.2.4: The diagrams on the left represent two equivalent free body diagrams for the same physical system with a single point force. The diagrams on the right represent two equivalent free body diagrams for the same physical system with a single surface force.

Video 1.2.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/8MR_w3ZOOiM.

1.3: Moments

A **moment** (also sometimes called a torque) is defined as the "tendency of a force to rotate a body". Where forces cause linear accelerations, moments cause **angular accelerations**. In this way moments, can be thought of as twisting forces.

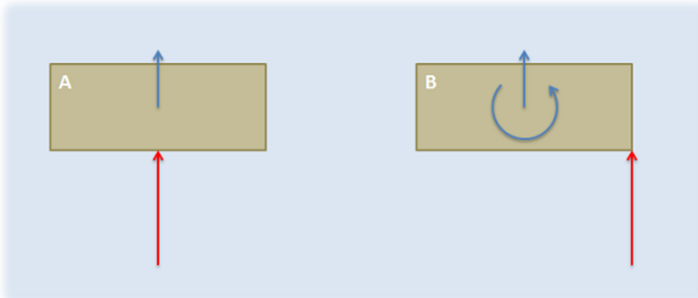


Figure 1.3.1: Imagine two boxes on an icy surface. The force on box A would simply cause the box to begin accelerating, but the force on box B would cause the box to both accelerate and to begin to rotate. The force on box B is exerting a moment, where the force on box A is not.

The Vector Representation of a Moment:

Moments, like forces, can be represented as vectors and have a magnitude, a direction, and a "point of application". For moments however a better name for the point of application is the **axis of rotation**. This will be the point or axis about which we will determine all the moments.

Magnitude:

The magnitude of a moment is the degree to which the moment will cause angular acceleration in the body it is acting on. It is represented by a scalar (a single number). The magnitude of the moment can be thought of as the strength of the twisting force exerted on the body. When a moment is represented as a vector, the magnitude of the moment is usually explicitly labeled. though the length of the moment vector also often corresponds to the relative magnitude of the moment.

The magnitude of the moment is measured in units of force times distance. The standard metric units for the magnitude of moments are Newton-meters, and the standard English units for a moment are foot-pounds.

$$M = F * d \tag{1.3.1}$$

$$\text{Metric: } N * m \tag{1.3.2}$$

$$\text{English: } lb * ft \tag{1.3.3}$$

Direction:

In a two-dimensional problem, the direction can be thought of as a scalar quantity corresponding to the direction of rotation the moment would cause. A moment that would cause a counterclockwise rotation is a positive moment, and a moment that would cause a clockwise rotation is a negative moment.

In a three-dimensional problem, however, a body can rotate about an axis in any direction. If this is the case, we need a vector to represent the direction of the moment. The direction of the moment vector will line up with the axis of rotation that moment would cause, but to determine which of the two directions we can use along that axis we have available we use the right hand rule. To use the right hand rule, align your right hand as shown in Figure $1.3.2$ so that your thumb lines up with the axis of rotation for the moment and your curled fingers point in the direction of rotation for your moment. If you do this, your thumb will be pointing in the direction of the moment vector.

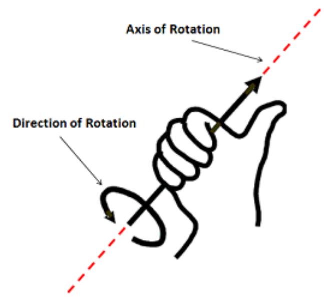


Figure 1.3.2: To use the right hand rule, align your right hand so that your thumb lines up with the axis of rotation for the moment and your curled fingers point in the direction of rotation for your moment. This means your thumb will be pointing in the direction of the moment vector.

If we look back to two-dimensional problems, all rotations occur about an axis pointing directly into or out of the page (the z-axis). Using the right hand rule, counterclockwise rotations are represented by a vector in the positive z direction and clockwise rotations are represented by a vector in the negative z direction.

Axis of Rotation:

In engineering statics problems we can choose any point/axis as the axis of rotation. However, the choice of this point will affect the magnitude and direction of the resulting moment, and the moment is only valid about that point.

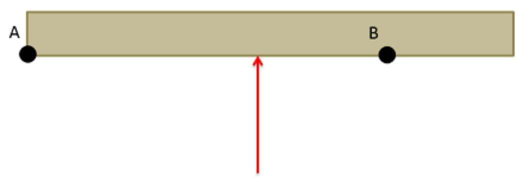


Figure 1.3.3: The magnitude and direction of a moment depends upon the chosen axis of rotation. For example, the single force above would cause different moments about Point A and Point B, because it would cause different rotations depending on the point we fix in place.

Though we can take the moment about any point in a statics problem, if we are **adding** together the moments from multiple forces, all the moments must be taken about a **common axis of rotation**. Moments taken about different points cannot be added together to find a "net moment."

Additionally, if we move into the subject of dynamics, where bodies are moving, we will want to relate moments to angular accelerations. For this to work, either we will need to take the moments about a single point that does not move (such as the hinge on a door) or we will need to take the moments about the center of mass of the body. Summing moments about other axes of rotation will not result in valid calculations.

Calculating Moments:

To calculate the moment that a force exerts on a body, we will have two main options: **scalar methods** and **vector methods**. Scalar methods are generally faster for two-dimensional problems where a body can only rotate clockwise or counterclockwise, while vector methods are generally faster for three-dimensional problems where the axis of rotation is more complex.

Video 1.3.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/RyOwVvYEFHU.

1.4: Free Body Diagrams

A free body diagram is a tool used to solve engineering mechanics problems. As the name suggests, the purpose of the diagram is to "free" the body from all other objects and surfaces around it so that it can be studied in isolation. We will also draw in any forces or moments acting on the body, including those forces and moments exerted by the surrounding bodies and surfaces that we removed.

The diagram below shows a ladder supporting a person and the free body diagram of that ladder. As you can see, the ladder is separated from all other objects and all forces acting on the ladder are drawn in with the key dimensions and angles shown.

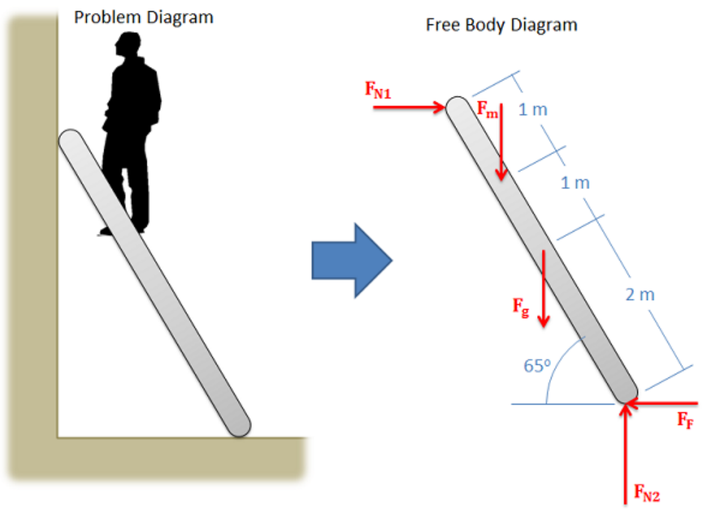


Figure 1.4.1: A ladder with a man standing on it is shown on the left. Assuming friction only at the base, a free body diagram of the ladder is shown on the right.

Constructing the Free Body Diagram:

The first step in solving most mechanics problems will be to construct a free body diagram. This simplified diagram will allow us to more easily write out the equilibrium equations for statics or strengths of materials problems, or the equations of motion for dynamics problems.

To construct the diagram we will use the following process:

1. First draw the body being analyzed, separated from all other surrounding bodies and surfaces. Pay close attention to the boundary, identifying what is part of the body, and what is part of the surroundings.
2. Second, draw in all **external** forces and moments acting directly on the body. Do not include any forces or moments that do not directly act on the body being analyzed. Do not include any forces that are **internal** to the body being analyzed.
3. Once the forces are identified and added to the free body diagram, the last step is to label any key dimensions and angles on the diagram.

Some common types of forces seen in mechanics problems are:

- **Gravitational Forces:** Unless otherwise noted, the mass of an object will result in a gravitational weight force applied to that body. This weight is usually given in pounds in the English system, and is modeled as 9.81 (g) times the mass of the body in kilograms for the metric system (resulting in a weight in Newtons). This force will always point down towards the center of the earth and act on the center of mass of the body.

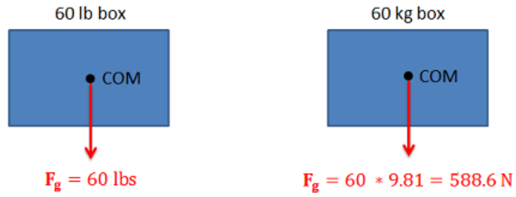


Figure 1.4.2: Gravitational forces always act downward on the center of mass.

- **Normal Forces (or Reaction Forces):** Every object in direct contact with the body will exert a normal force on that body which prevents the two objects from occupying the same space at the same time. Note that only objects in direct contact can exert normal forces on the body.
  - An object in contact with another object or surface will experience a normal force that is perpendicular (normal) to the surfaces in contact.
  - Joints or connections between bodies can also cause reaction forces or moments, and we will have one force or moment for each type of motion or rotation the connection prevents.

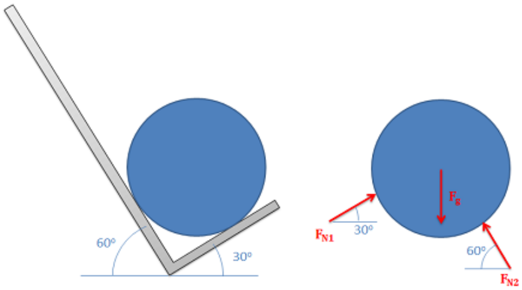


Figure 1.4.3: Normal forces always act perpendicular to the surfaces in contact. The barrel in the hand truck shown on the left has a normal force at each contact point.

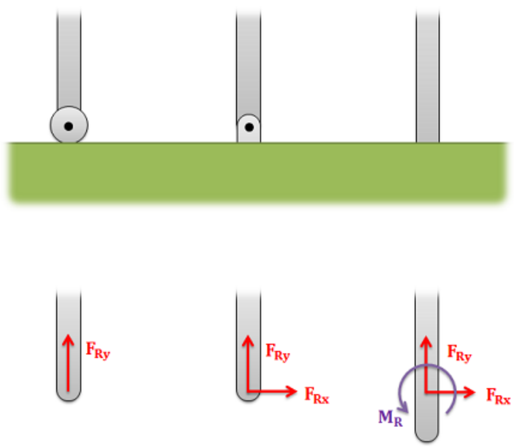


Figure 1.4.4: The roller on the left allows for rotation and movement along the surface, but a normal force in the y direction prevents motion vertically. The pin joint in the center allows for rotation, but normal forces in the x and y directions prevent motion in all directions. The fixed connection on the right has a normal forces preventing motion in all directions and a reaction moment preventing rotation.

- **Friction Forces:** Objects in direct contact with the body can also exert friction forces, which will resist the two bodies sliding against one another, on the body. These forces will always be perpendicular to the surfaces in contact. Friction is the subject of an entire chapter in this book, but for simple scenarios we usually assume rough or smooth surfaces.
  - For smooth surfaces we assume that there is no friction force.
  - For rough surfaces we assume that the bodies will not slide relative to one another, no matter what. In this case, the friction force is always just large enough to prevent this sliding.

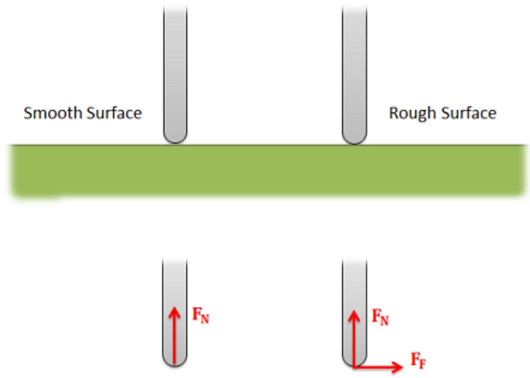


Figure 1.4.5: For a smooth surface we assume only a normal force perpendicular to the surface. For a rough surface we assume normal and friction forces are present.

- **Tension in Cables:** Cables, wires or ropes attached to the body will exert a tension force on the body in the direction of the cable. These forces will always pull on the body, as ropes, cables and other flexible tethers cannot be used for pushing.

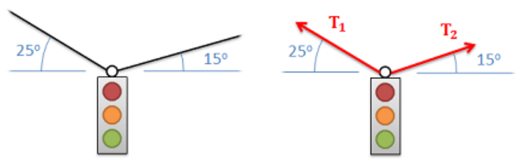


Figure 1.4.6: The tension force in cables always acts along the direction of the cable and will always be a pulling force.

The above forces are the most common, but other forces such as pressure from fluids, spring forces and magnetic forces exist and may act on the body.

Video 1.4.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/Kr7obGR68-Y.

Worked Problems:

Example 1.4.1

The drawing below shows two boxes sitting on a table. Draw a free body diagram of box A and box B.

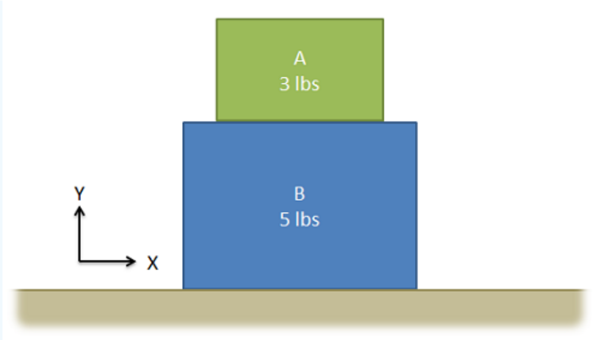


Figure 1.4.7: problem diagram for Example 1.4.1; two boxes are stacked on a flat surface, with one weighing 3 lbs on top of another weighing 5 lbs.

Solution

Video 1.4.2: Worked solution to example problem 1.4.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/RMVa9kioALs.

Example 1.4.2

Two equally sized barrels are being transported in a handtruck as shown below. Draw a free body diagram of each of the two barrels.

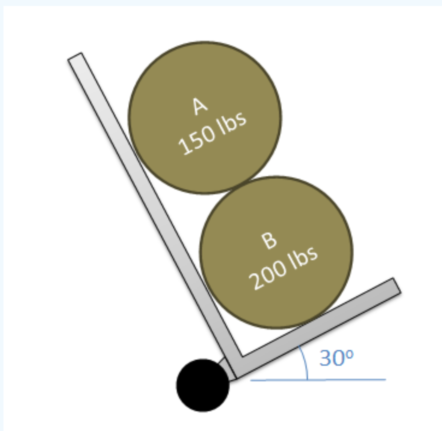


Figure 1.4.8: problem diagram for Example 1.4.2; two barrels are stacked horizontally, on a handcart tilted so the bottom is 30° above the horizontal.

Solution

Video 1.4.3: Worked solution to example problem 1.4.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/1a9gjFOIpK8.

Example 1.4.3

The car shown below is moving and then slams on the brakes locking up all four wheels. The distance between the two wheels is 8 feet and the center of mass is 3 feet behind and 2.5 feet above the point of contact between the front wheel and the ground. Draw a free body diagram of the car as it comes to a stop.

Figure 1.4.9: Car traveling on a level surface, facing left. Public domain image, no author listed.

Solution

Video 1.4.4: Worked solution to example problem 1.4.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/GiB3_fSlJBA.

Example 1.4.4

A 600-pound load is supported by a 5 meter long, 100-pound cantilever beam. Assume the beam is firmly anchored to the wall. Draw a free body diagram of the beam.

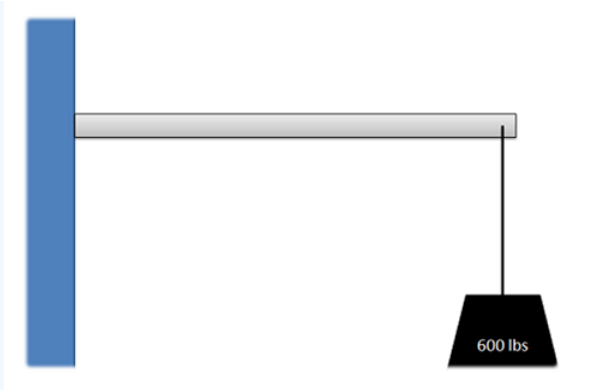


Figure 1.4.10: problem diagram for Example 1.4.4; a 600-lb load hangs from the free end of a horizontal beam whose other end is attached to a wall.

Solution

Example 1.4.5

The main arm of a crane has a mass of 400 kg (assume the center of mass is at the midpoint of the arm), and supports a 200 kg load and a 600 kg counterweight. The arm is connected to the vertical support via a pin joint and two flexible cables. Draw a free body diagram of the arm.

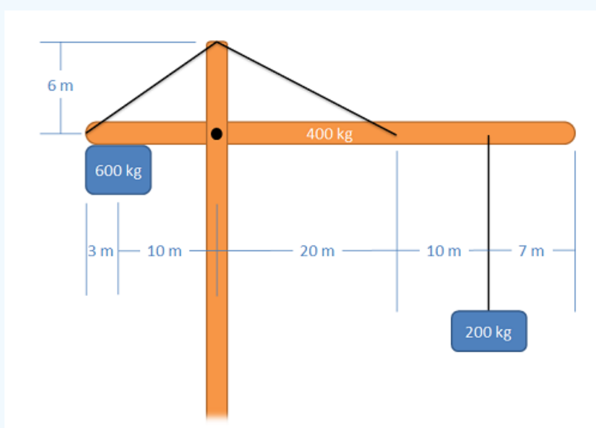


Figure 1.4.11: problem diagram for Example 1.4.5; a crane's arm, currently in the horizontal position, holds a load and a counterweight on opposite ends.

Solution

Video 1.4.6: Worked solution to example problem 1.4.5, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/0V7ULRnnhmA.

1.5: Newton's First Law

Newton's first law states that: **"A body at rest will remain at rest unless acted on by an unbalanced force. A body in motion continues in motion with the same speed and in the same direction unless acted upon by an unbalanced force."**

This law, also sometimes called the "law of inertia", means that bodies maintain their current velocity unless a force is applied to change that velocity. If an object is at rest with zero velocity it will remain at rest until some force begins to change that velocity, and if an object is moving at a set speed and in a set direction it will remain at that same velocity until some force acts on it to change its velocity.

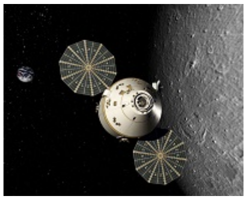


Figure 1.5.1: In the absence of friction in space, this space capsule will maintain its current velocity until some outside force causes that velocity to change. Public Domain image by NASA.

Figure 1.5.2: This rock is at rest with zero velocity and will remain at rest until a net force causes the rock to move. The net force on the rock is the sum of any force pushing the rock and the friction force of the ground on the rock opposing that force. Image by Liz Gray CC-BY-SA 2.0.

Net Forces:

It is important to note that the **net force** is what will cause a change in velocity. The net force is the sum of all forces acting on the body. For example, we can imagine gently pushing on the rock in the figure above and observing that the rock does not move. This is because we will have a friction force equal in magnitude and opposite in direction opposing our gentle pushing force. The sum of these two forces will be equal to zero, therefore the net force is zero and the change in velocity is zero.

Rotational Motion:

Newton's first law also applies to moments and rotational velocities. A body will maintain it's current rotational velocity until a net moment is exerted to change that rotational velocity. This can be seen in things like toy tops, flywheels, stationary bikes, and other objects that will continue spinning once started until brakes or friction stop them.

Figure 1.5.3: In the absence of friction, this spinning top would continue to spin forever, but the small frictional moment exerted at the point of contact between the top and the ground will slow the tops spinning over time. Image by Carrotmadman6 CC-BY-2.0.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/MeY-Cj93Tm4.

1.6: Newton's Second Law

Translational Motion:

Newton's second law states that: **"When a net force acts on any body with mass, it produces an acceleration of that body. The net force will be equal to the mass of the body times the acceleration of the body."**

$$\vec{F} = m\vec{a} \tag{1.6.1}$$

You will notice that the force and the acceleration in the equation above have an arrow above them. This means that they are vector quantities, having both a magnitude and a direction. Mass, on the other hand, is a scalar quantity having only a magnitude. Based on the above equation, you can infer that the magnitude of the net force acting on the body will be equal to the mass of the body times the magnitude of the acceleration, and that the direction of the net force on the body will be equal to the direction of the acceleration of the body.

Rotational Motion:

Newton's second law also applies to moments and rotational velocities. The revised version of the second law equation states that the net moment acting on the object will be equal to the mass moment of inertia of the body about the axis of rotation (I) times the angular acceleration of the body.

$$\vec{M} = I * \vec{\alpha} \tag{1.6.2}$$

You should again notice that the moment and the angular acceleration of the body have arrows above them, indicating that they are vector quantities with both a magnitude and direction. The mass moment of inertia, on the other hand, is a scalar quantity having only a magnitude. The magnitude of the net moment will be equal to the mass moment of inertia times the magnitude of the angular acceleration, and the direction of the net moment will be equal to the direction of the angular acceleration.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/3PF2uNGW7Dw.

1.7: Newton's Third Law

Newton's Third Law states **"For any action, there is an equal and opposite reaction."** By "action" Newton meant a force, so for every force one body exerts on another body, that second body exerts a force of equal magnitude but opposite direction back on the first body. Since all forces are exerted by bodies (either directly or indirectly), all forces come in pairs, one acting on each of the bodies interacting.

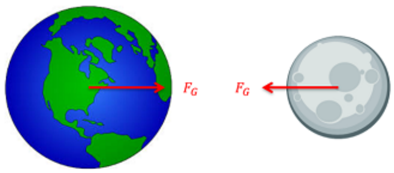


Figure 1.7.1: The gravitational pull of the Earth and Moon represent a Newton's Third Law pair. The Earth exerts a gravitational pull on the Moon, and the Moon exerts an equal and opposite pull on the Earth. Image adapted from Public Domain images, no authors listed.

Though there may be two equal and opposite forces acting on a single body, it is important to remember that for each of the forces a Third Law pair acts on a separate body. This can sometimes be confusing when there are multiple Third Law pairs at work. Below are some examples of situations where multiple Third Law pairs occur.

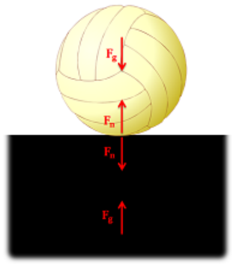


Figure 1.7.2: This volleyball resting on a surface has two pairs of Third Law forces. The first consists of the gravitational forces (one force on the ball and one force on the ground). The second consists of the normal forces at the point of contact (one force on the ball and one force on the ground). Image adapted from Public Domain image, no author listed.

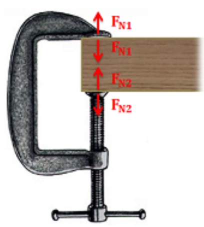


Figure 1.7.3: If we ignore the weight of the two objects, this clamp will also have two pairs of Third Law forces. The first will be a set of normal forces at the top point of contact (one force on the wood and one force on the clamp) and the second will be another set of normal forces at the bottom point of contact (one force on the wood and one force on the clamp) Image adapted from Public Domain image, no author listed.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/zB5l95jwKr4.

1.8: Chapter 1 Homework Problems

Exercise 1.8.1

A pulley system is being used to hoist a 50 kg engine block as shown below. If distance d is currently 1 meter and we assume the pulleys are all frictionless, draw a free body diagram of the engine block with the attached pulley. Include all forces and important angles.

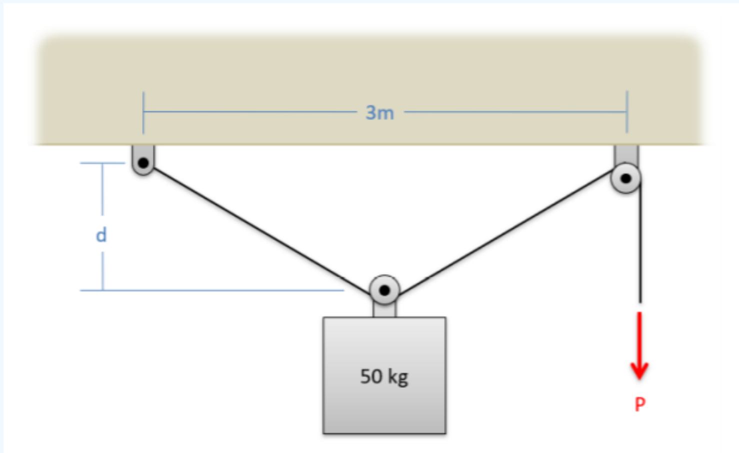


Figure 1.8.1: problem diagram for Exercise 1.8.1; an engine block is suspended by a cable with one end attached to an anchor point and the other end passing over a pulley.

Exercise 1.8.2

The car shown below has a weight of 4500 lbs and a center of mass at point G. Assuming the car is not moving and is sitting on a level surface, draw a free body diagram of the car. Include all forces and important distances.

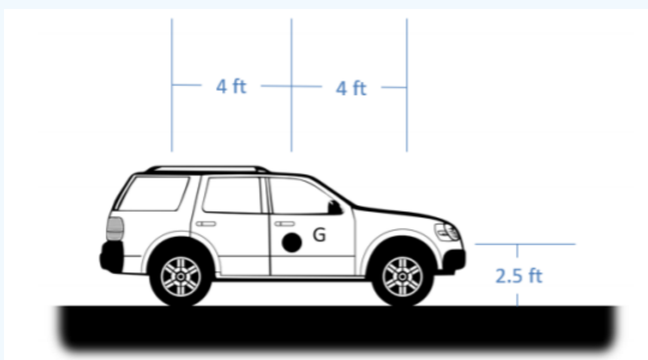


Figure 1.8.2: problem diagram for Exercise 1.8.2; a car sitting on a level surface is marked with a point G indicating the location of its center of mass.

Exercise 1.8.3

A telephone pole sits on a rough surface. A cable attached to an excavator is then used to pull the pole along the surface as shown below. Assume the telephone pole has a mass of 350 kg and a length of 12 meters. Draw a free body diagram of the telephone pole. Include all forces, important distances, and important angles.

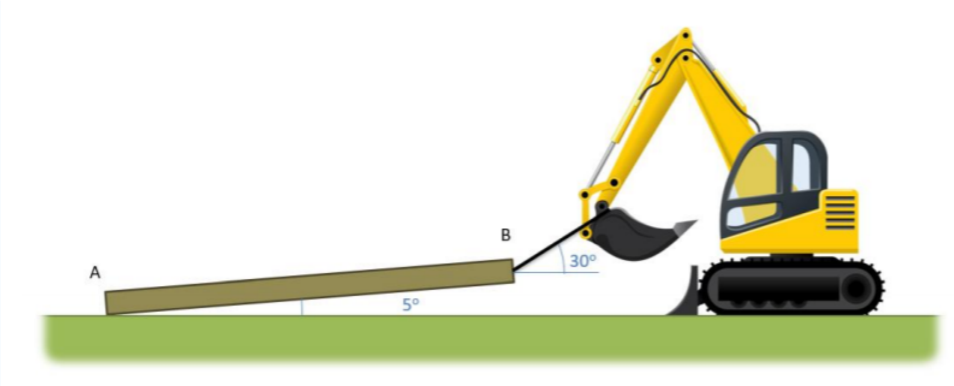


Figure 1.8.3: problem diagram for Exercise 1.8.3; an excavator pulls a telephone pole across the ground at an angle, through the use of a cable.

CHAPTER OVERVIEW

2: Static Equilibrium in Concurrent Force Systems

2.0: Chapter 2 Video Introduction

Video introduction to Chapter 2, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/VXhvhP8VBMY.

2.1: Static Equilibrium

Objects in static equilibrium are objects that are not accelerating (either linear acceleration or angular acceleration). These objects may be stationary, such as a building or a bridge, or they may have a constant velocity, such as a car or truck moving at a constant speed on a straight patch of road.

Figure 2.1.1: (left) Because this high rise building is stationary with no acceleration, the members and overall structure are in equilibrium. Image by Jakembradford CC-BY-SA 4.0. (right): Assuming that this truck is maintaining a constant speed and direction, this truck is in equilibrium because its velocity is not changing over time. Public Domain image by Klever.

Newton's Second Law states that the force exerted on an object is equal to the mass of the object times the acceleration it experiences. Therefore, if we know that the acceleration of an object is equal to zero, then we can assume that the sum of all forces acting on the object is zero. Individual forces acting on the object, represented by force vectors, may not have zero magnitude but the sum of all the force vectors will always be equal to zero for objects in equilibrium. Engineering statics is the study of objects in static equilibrium, and the simple assumption of all forces adding up to zero is the basis for the subject area of engineering statics.

$$\sum \vec{F} = m\vec{a} \tag{2.1.1}$$

$$a = 0\,; \; \sum \vec{F} = 0 \tag{2.1.2}$$

Equilibrium follows a similar pattern for angular accelerations. The rotational equivalent of Newton's Second Law states that the moment exerted on an object is equal to the moment of inertia of that object times the angular acceleration of the object. If we know the angular acceleration of an object is equal to zero, then we know the sum of all moments acting on the object is equal to zero.

$$\sum \vec{M} = I\vec{\alpha} \tag{2.1.3}$$

$$\vec{\alpha} = 0\,; \; \sum \vec{M} = 0 \tag{2.1.4}$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/2Ekp8MkgUYI.

2.2: Point Forces as Vectors

A point force is any force where the **point of application** is considered to be a **single point**. In reality, most forces are technically surface forces, where the force is applied over an area, but when the area is small enough (in comparison to the bodies being analyzed) it can often be approximated as a point force. Because point forces can be represented as a single vector (rather than a field of vectors for distributed forces), they are much easier to work with in engineering analysis. For this reason, point forces are used in place of distributed forces in engineering analysis whenever possible. Below are some examples of where it is appropriate to use point forces.

Figure 2.2.1: The tensions in the cables supporting this container can be treated as point forces pulling in the direction of the cables. Adapted from Image by maxronnersjo CC-BY-SA 3.0.

Figure 2.2.2: The friction force between the bow and string on this cello can be treated as a point force. Adapted from Public Domain image by Levi.

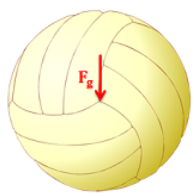


Figure 2.2.3: Though gravitational forces are technically body forces, they are often approximated as a single point force acting on the center of gravity of the object. Adapted from Public Domain image, no author listed.

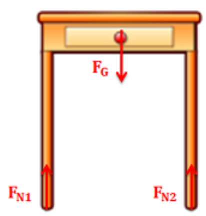


Figure 2.2.4: The gravitational force and the normal forces acting on each leg of this table can all be approximated as point forces. Adapted from Public Domain image by Seahen.

In addition to the magnitude, direction, and point of application of the point force, another important term to understand is the **line of action** of the force. The line of action of a force is the line along which the force acts. Given the direction and point of application, one can find the line of action, but this term will be important in discussing concurrent forces and in the principle of transmissibility.

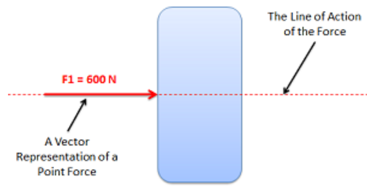


Figure 2.2.5: The line of action of a point force is the line along which the force acts.

Force Vector Representation:

When vectors are drawn to form free body diagrams, the magnitude and direction are usually given in one of two formats:

- Overall magnitude and angle(s) to indicate direction (often called **magnitude and direction form**).
- Magnitudes in each of the coordinate directions (often called **component form**).

In either format we will need two values to fully define a force vector in a 2D system (either a magnitude and a single angle or a magnitude in each of the two coordinate axes), and three values to fully define a force vector in a 3D system (either a magnitude and two angles or a magnitude in each of the three coordinate axes). Below are some examples of force vectors in both representations.

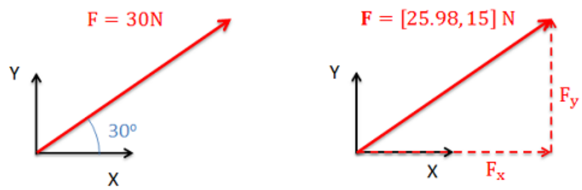


Figure 2.2.6: The same force can be represented with a magnitude and an angle, as shown in the left, or with magnitudes in relation to each of the coordinate axes as shown on the right.

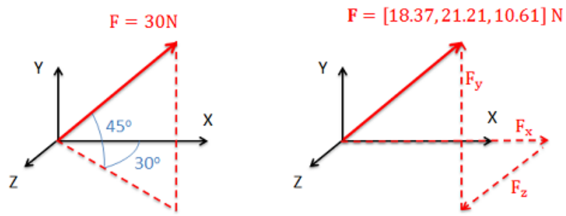


Figure 2.2.7: In three dimensions forces are represented with either a magnitude and two directions, as shown on the left, or with magnitudes in relation to each of the three coordinate axes as shown on the right.

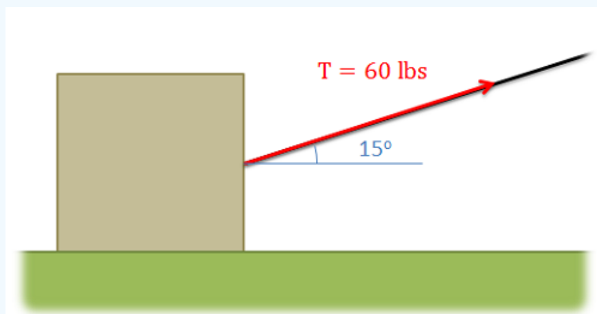
Changing Force Vector Forms:

Because the two different forms of the vector are equivalent, we can switch between representations without changing the problem. Often in engineering problems, it will initially be easier to write the force in magnitude and angle form, but later, analysis will be easier if forces are written in component form. To switch from magnitude and direction form to component form you will use right triangles and trigonometry to determine the component of the overall magnitude in each direction. This is a simple vector decomposition, and more information on this process can be seen on the vector decomposition page. To switch back from component form into magnitude and direction form you simply use the reverse of this initial process.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/kq8vqOKhKeA.

Example 2.2.1

The tension force on the box below is given in magnitude and direction form. Redraw the diagram with the tension force given in component form.

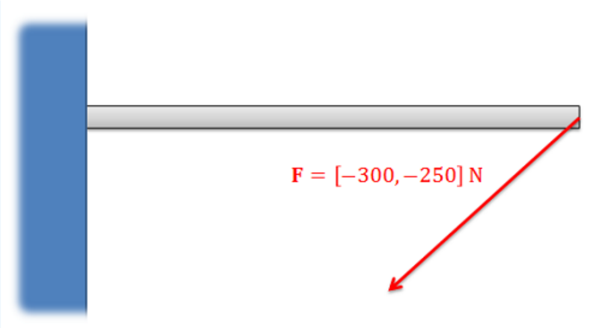


Solution

Video 2.2.2: Worked solution to example problem 2.2.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/ZERURXWnlDg.

Example 2.2.2

The force acting on the cantilever beam shown below is given in component form. Redraw the diagram with the force given in magnitude and direction form.

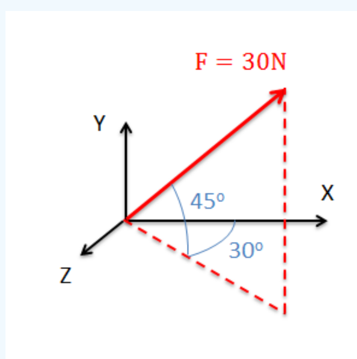


Solution

Video 2.2.3: Worked solution to example problem 2.2.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/M3UjDfZzRHY.

Example 2.2.3

The force shown below is given in magnitude and direction form. Redraw the diagram with the force vector given in component form.



Solution

Video 2.2.4: Worked solution to example problem 2.2.3, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/DroNv0TxnyA.

2.3: Principle of Transmissibility

The principle of transmissibility states that the point of application of a force can be moved anywhere along its line of action without changing the **external reaction forces** on a **rigid body**. Any force that has the same magnitude and direction, and which has a point of application somewhere along the same line of action will cause the same acceleration and will result in the same moment. Therefore, the points of application of forces may be moved along the line of action to simplify the analysis of rigid bodies.

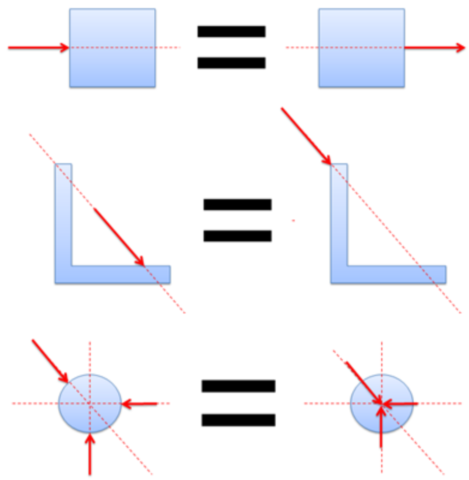


Figure 2.3.1: Because of the principle of transmissibility, each of the above pairs is equivalent.

When analyzing the internal forces (stress) in a rigid body, the exact point of application does matter. This difference in stresses may also result in changes in geometry which will in turn affect reaction forces. For this reason, the principle of transmissibility should only be used when examining external forces on bodies that are assumed to be rigid.

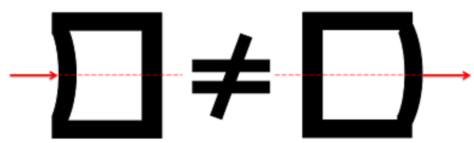


Figure 2.3.2: The exact point of application of a force will impact how internal forces (stresses) are distributed, so the principle of transmissibility cannot be applied when examining internal forces.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/sx__xzA7eqM.

history is available upon request.

2.4: Concurrent Forces

A set of point forces is considered concurrent if all the **lines of action of those forces all come together at a single point**.

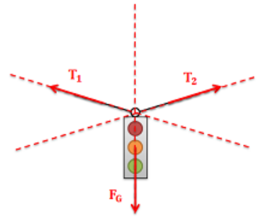


Figure 2.4.1: Because the lines of action for the gravitational force and the two tension forces line up at a single point, these forces are considered concurrent.

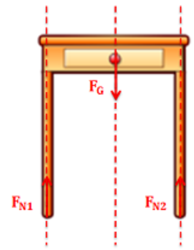


Figure 2.4.2: Because the lines of action of the gravitational force and the two normal forces do not intersect at a single point, these forces are not considered concurrent. Adapted from Public Domain image by Seahen.

Because the forces all act through a single point, there are no moments about this point. Because no moments exist, we can treat this body as a **particle**. In fact, because real particles only exist in theory, most particle analysis is actually applied to extended bodies with concurrent forces acting on them.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/puLnyApKfuc.

2.5: Equilibrium Analysis for Concurrent Force Systems

If a body is in static equilibrium, then by definition that body is not accelerating. If we know that the body is not accelerating then we know that **the sum of the forces acting on that body must be equal to zero**. This is the basis for equilibrium analysis for a particle.

In order to solve for any unknowns in our sum of forces equation, we actually need to turn the one vector equation into a set of scalar equations. For two dimensional problems, we will split our one vector equation down into two scalar equations. We do this by summing up all the x components of the force vectors and setting them equal to zero in our first equation, and summing up all the y components of the force vectors and setting them equal to zero in our second equation.

$$\sum \vec{F} = 0 \tag{2.5.1}$$

$$\sum F_x = 0\,; \;\sum F_y = 0 \tag{2.5.2}$$

We do something similar in three dimensional problems except we will break all our force vectors down into x, y, and z components, setting the sum of x components equal to zero for our first equation, the sum of all the y components equal to zero for our second equation, and the sum of all our z components equal to zero for our third equation.

$$\sum \vec{F} = 0 \tag{2.5.3}$$

$$\sum F_x = 0\,; \;\sum F_y = 0\,; \;\sum F_z = 0 \tag{2.5.4}$$

Once we have written out the equilibrium equations, we can solve the equations for any unknown forces.

Finding the Equilibrium Equations:

The first step in finding the equilibrium equations is to **draw a free body diagram** of the body being analyzed. This diagram should show all the known and unknown force vectors acting on the body. In the free body diagram, provide values for any of the know magnitudes or directions for the force vectors and provide variable names for any unknowns (either magnitudes or directions).

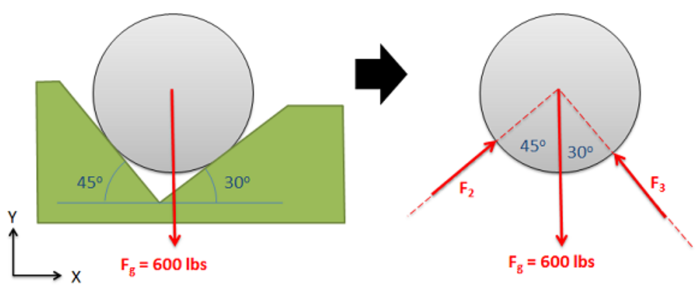


Figure 2.5.1: The first step in equilibrium analysis is drawing a free body diagram. This is done by removing everything but the body and drawing in all forces acting on the body. It is also useful to label all forces, key dimensions, and angles.

Next you will need to chose the x, y, and z axes. These axes do need to be perpendicular to one another, but they do not necessarily have to be horizontal or vertical. If you choose coordinate axes that line up with some of your force vectors you will simplify later analysis.

Once you have chosen axes, you need to break down all of the force vectors into components along the x, y and z directions (see the vectors page in Appendix 1 if you need more guidance on this). Your first equation will be the sum of the magnitudes of the components in the x direction being equal to zero, the second equation will be the sum of the magnitudes of the components in the y direction being equal to zero, and the third (if you have a 3D problem) will be the sum of the magnitudes in the z direction being equal to zero. Collectively these are known as the **equilibrium equations**.

Once you have your equilibrium equations, you can solve them for unknowns using algebra. The number of unknowns that you will be able to solve for will be the number of equilibrium equations that you have. In instances where you have more unknowns than equations, the problem is known as a **statically indeterminate problem** and you will need additional information to solve for the given unknowns.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/Dbd9SvdfoN8.

Example 2.5.1

The diagram below shows a 3-lb box (Box A) sitting on top of a 5-lb box (box B). Determine the magnitude and direction of all the forces acting on box B.

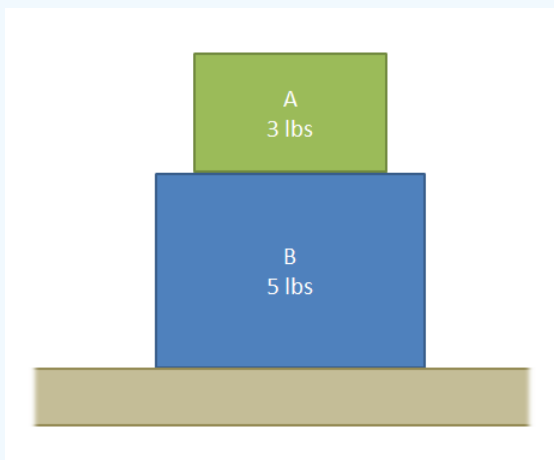


Figure \(\
PageIndex
{2}\): problem diagram for Example \(\
PageIndex
{1}\); two stacked boxes sitting on a flat surface.

Solution

Video 2.5.2: Worked solution to example problem 2.5.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/J54OZSitzzM.

Example 2.5.2

A 600-lb barrel rests in a trough as shown below. The barrel is supported by two normal forces (F_2 and F_3). Determine the magnitude of both of these normal forces.

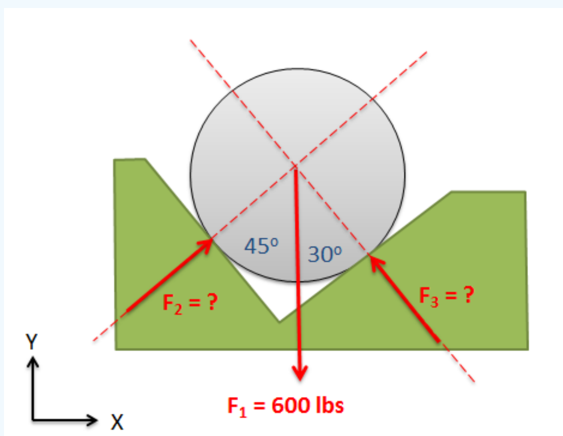


Figure 2.5.3: problem diagram for Example 2.5.2; a barrel resting in a trough with straight, angled sides.

Solution

Video 2.5.3: Worked solution to example problem 2.5.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/qKhZvf55Bc0.

Example 2.5.3

A 6-kg traffic light is supported by two cables as shown below. Find the tension in each of the cables supporting the traffic light.

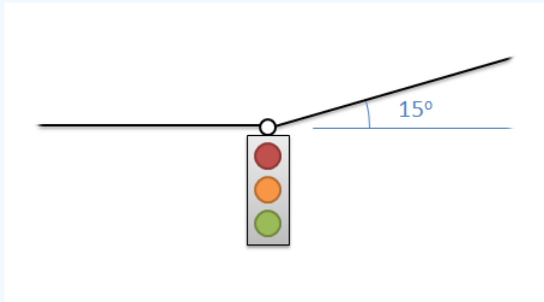


Figure 2.5.4: problem diagram for Example 2.5.3; a traffic light is held in midair by two cables, one horizontal and one angled..

Solution

Video 2.5.4: Worked solution to example problem 2.5.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Oi2yDg1SmrI.

Example 2.5.4

A 400-kg wrecking ball rests against a surface as shown below. Assuming the wrecking ball is currently in equilibrium, determine the tension force in the cable supporting the wrecking ball and the normal force that exists between the wrecking ball and the surface.

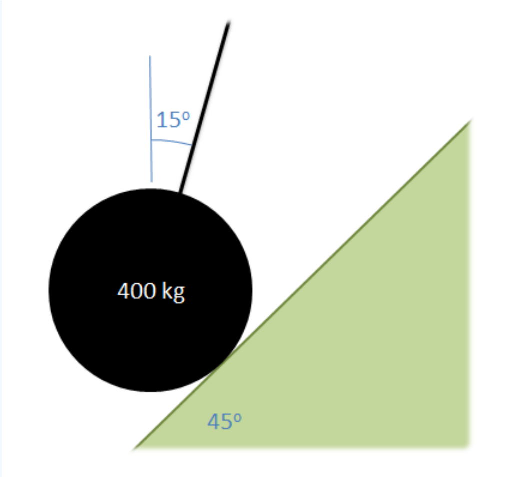


Figure 2.5.5: problem diagram for Example 2.5.4; a wrecking ball on a cable is resting against an angled surface.

Solution

Video 2.5.5: Worked solution to example problem 2.5.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/gETMTfy5Sew.

Example 2.5.5

Barrels A and B are supported in a foot truck as seen below. Assuming the barrels are in equilibrium, determine all forces acting on barrel B.

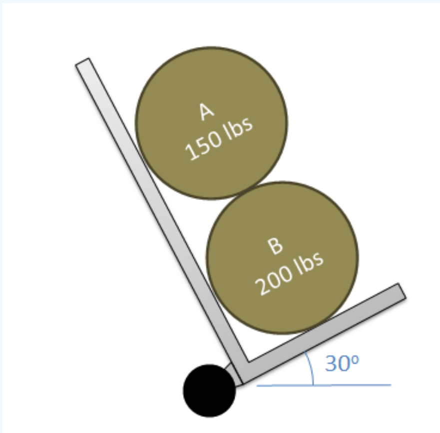


Figure 2.5.6: problem diagram for Example 2.5.5; two barrels stacked on their sides are on a handcart, whose bottom is tilted upwards.

Solution

Video 2.5.6: Worked solution to example problem 2.5.5, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/8DgrClhT4AM.

Example 2.5.6

Three soda cans, each weighing 0.75 lbs and having a diameter of 4 inches, are stacked in a formation as shown below. Assuming no friction forces, determine the normal forces acting on can B.

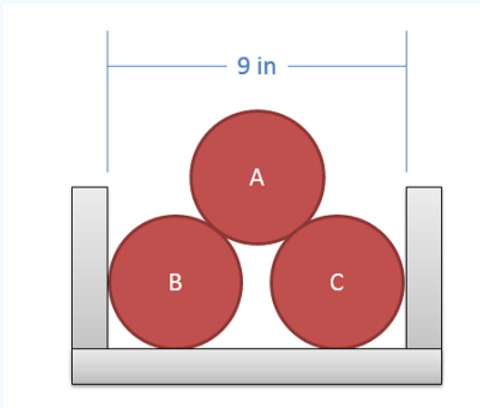


Figure 2.5.7: problem diagram for Example 2.5.6; three soda cans are stacked lying on their sides, in a flat area bounded on two sides by walls.

Solution

Video 2.5.7: Worked solution to example problem 2.5.6, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/lAUahV7Mml4.

Example 2.5.7

The skycam shown below is supported by three cables. Assuming the skycam has a mass of 20 kg and that it is currently in a state of equilibrium, find the tension in each of the three cables supporting the skycam.

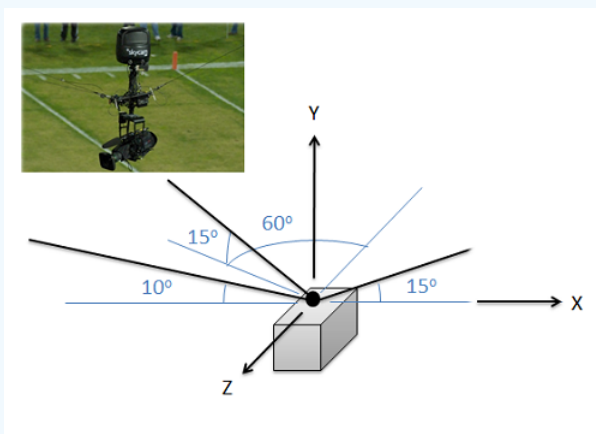


Figure 2.5.8: problem diagram for Example 2.5.7; a skycam is held in midair by 3 cables, whose angles in relation to a three-dimensional coordinate plane are shown. Image by Jrienstra CC-BY-SA 3.0.

Solution

Video 2.5.8: Worked solution to example problem 2.5.7, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/FD3yKyfXkGU.

Example 2.5.8

A hot air balloon is tethered to the ground with three cables as shown below. If the balloon is pulling upwards with a force of 900 lbs, what is the tension in each of the three cables?

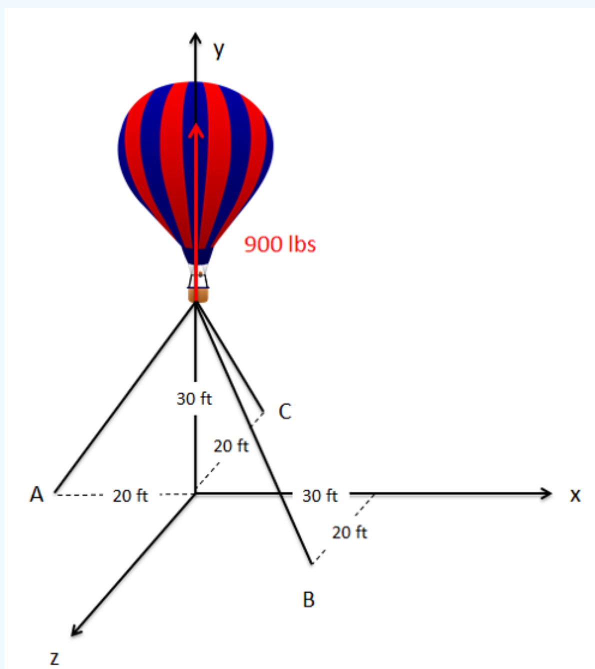


Figure 2.5.9: problem diagram for Example 2.5.8; a hot-air balloon is tethered to the ground by 3 cables, whose points of contact with the ground are given in relation to a three-dimensional coordinate plane. Adapted from image by L. Aragon CC-BY-SA 3.0.

Solution

Video 2.5.9: Worked solution to example problem 2.5.8, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/HQqNGJR3ybQ.

2.6: Chapter 2 Homework Problems

Exercise 2.6.1

A 30 kg barrel is sitting on a handcart as shown below. Determine the normal forces at A and B.

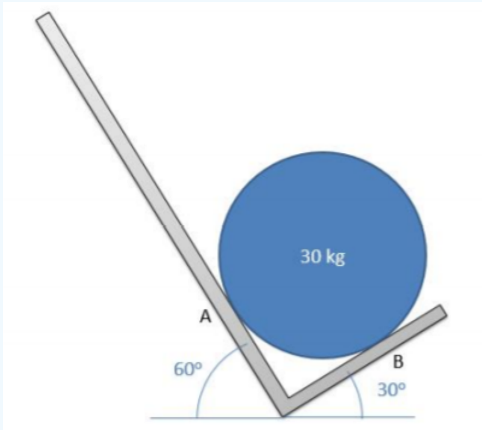


Figure 2.6.1: problem diagram for Exercise 2.6.1; a barrel sitting in a tilted handcart.

Answer

$F_A = 147.15\ N;\ F_B = 254.87\ N$

Exercise 2.6.2

A 0.25kg ball rolls into a corner as shown below. Assuming the surfaces are smooth (no friction), determine the normal forces at A and B.

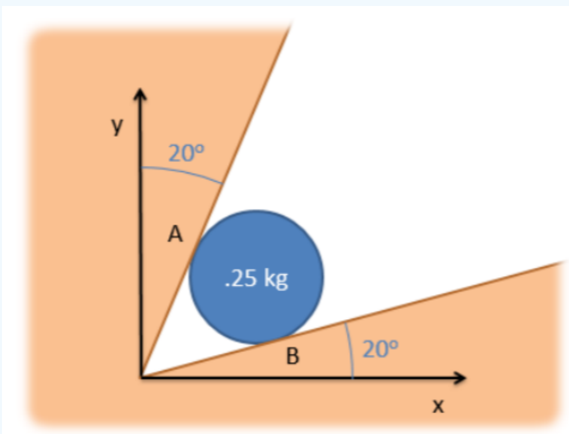


Figure 2.6.2: problem diagram for Exercise 2.6.2; a ball wedged into a narrow corner.

Answer

$F_A = 1.09\ N;\ F_B = 3.01\ N$

Exercise 2.6.3

A traffic light is supported by two cables as shown below. The tension in cable one is measured to be 294.8 N. What is the tension in cable two? What is the mass of the traffic light?

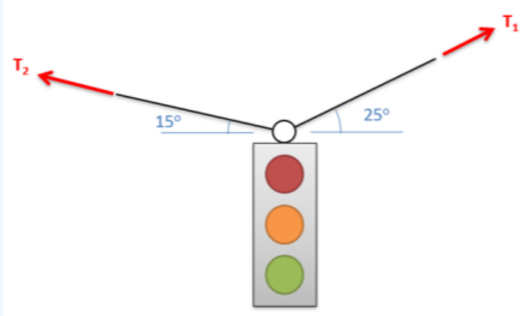


Figure 2.6.3: problem diagram for Exercise 2.6.3; a traffic light suspended by two angled cables.

Answer

$T_2 = 276.6\ N;\ m = 20\ kg$

Exercise 2.6.4

A 50 kg truck engine is lifted using the setup shown below. Assuming that the pulleys shown in the diagram are frictionless, what force P must be applied to the cable to hold the engine in the position shown below with d = 1 meter? (Hint: Draw a free body diagram of the pulley supporting the engine block)

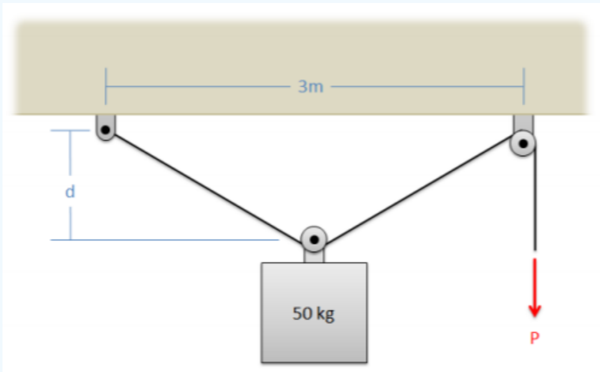


Figure 2.6.4: problem diagram for Exercise 2.6.4; an engine block suspended by a cable running through one anchor point and one pulley.

Answer

$P = 442.1\ N$

Exercise 2.6.5

Two weights are supported via cables as shown below. If body B has a weight of 60 pounds, what is the expected weight of body A based on the angles of the cables?

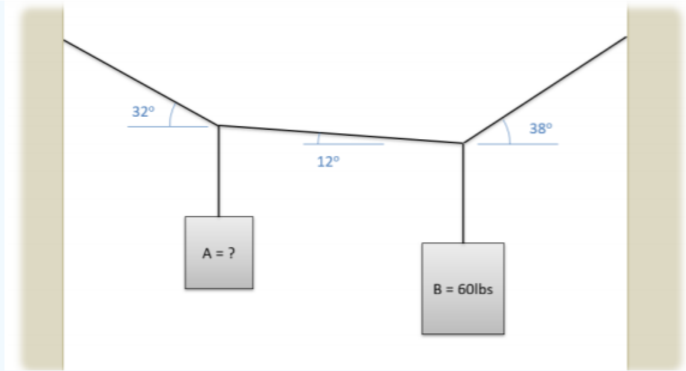


Figure 2.6.5: problem diagram for Exercise 2.6.5; two weights hanging from a single cable with fixed ends.

Answer

$F_{gA} = 24.89\ lbs$

Exercise 2.6.6

Three equally sized cylinders, each with mass 100 kg, are stacked in a groove as shown below. Determine all forces acting on cylinder C and show them in a diagram.

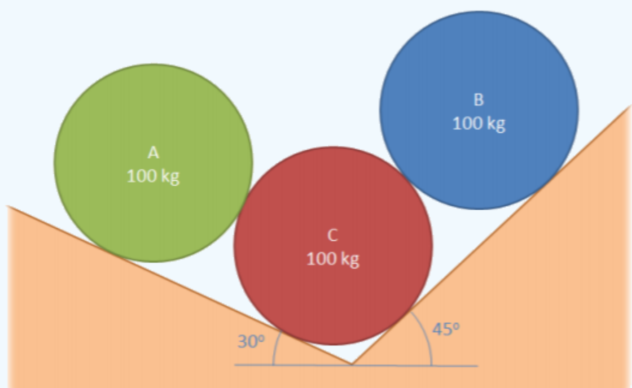


Figure 2.6.6: problem diagram for Exercise 2.6.6; three balls wedged in a groove with angled sides.

Answer

$F_{AC} = 490.5\ N;\ F_{BC} = 693.7\ N;\ F_{C1} = 1304.6\ N;\ F_{C2} = 829.7\ N;\ F_g = 981\ N$

Exercise 2.6.7

You are hanging a pterodactyl model from the ceiling of a museum with three cables as shown below. Assuming the pterodactyl model has a mass of 260 kg, what is the tension we would expect in each of the three cables?

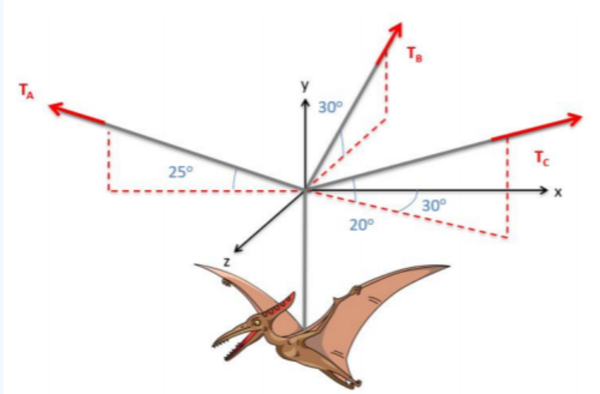


Figure 2.6.7: problem diagram for Exercise 2.6.7; a pterodactyl model hanging from the intersection of three unequally angled cables attached to the ceiling.

Answer

$T_A = 2306.94\,N$; $T_B = 1393.86\,N$; $T_C = 2569.19\,N$

Exercise 2.6.8

A hot air balloon is tethered as shown below. Assuming that the balloon is pulling upward with a force of 900 lbs, determine the tension in each of the cables.

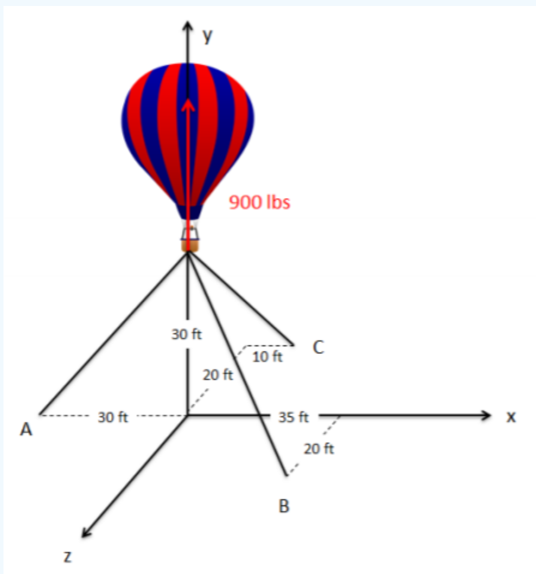


Figure 2.6.8: problem diagram for Exercise 2.6.8; a hot air balloon tethered to the ground by three unequally spaced cables.

Answer

$T_A = 545.5\,lbs$; $T_B = 430.7\,lbs$; $T_C = 320.7\,lbs$

CHAPTER OVERVIEW

3: Static Equilibrium in Rigid Body Systems

3.0: Video Introduction to Chapter 3

Video introduction to the topics covered in Chapter 3, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/x_L6S6ohu-k.

3.1: Moment of a Force about a Point (Scalar Calculation)

The **moment of a force** is the tendency of some forces to cause rotation. Any easy way to visualize the concept is set a box on smooth surface. If you were to apply a force to the center of the box, it would simply slide across the surface without rotating. If you were instead to push on one side of the box, it will start rotating as it moves. Even though the forces have the same magnitude and the same direction, they cause different reactions. This is because the off-center force has a different point of application, and exerts a moment about the center of the box, whereas the force on the center of the box does not exert a moment about the box's center point.

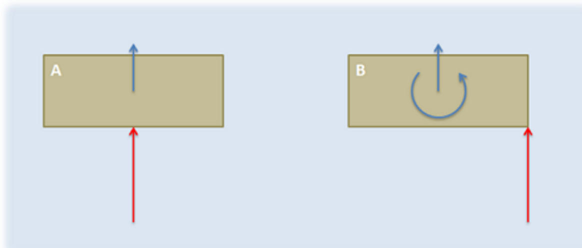


Figure 3.1.1: If we push a box in the center, it will simply begin sliding. If we push a box off-center, we will exert a moment and the box will rotate in addition to sliding.

Just like forces, moments have a magnitude (the degree of rotation it would cause) and a direction (the axis the body would rotate about). Determining the magnitude and direction of these moments about a given point is an important step in the analysis of rigid body systems (bodies that are both rigid and not experiencing concurrent forces). The scalar method below is the easiest way to do this in simple two-dimensional problems, while the alternative vector methods, which will be covered later, work best for more complex three-dimensional systems.

The Scalar Method in 2 Dimensions

In discussing how to calculate the moment of a force about a point via scalar quantities, we will begin with the example of a force on a simple lever as shown below. In this simple lever there is a force on the end of the lever, distance d away from the center of rotation for the lever (point A) where the force has a magnitude F.

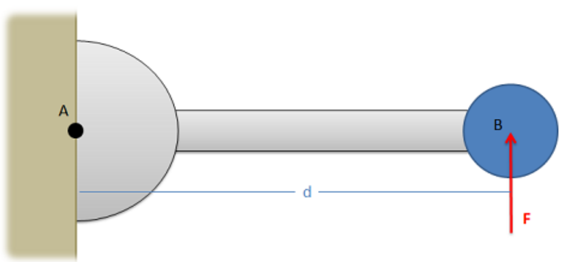


Figure 3.1.2: The magnitude of the moment that force F exerts about point A on this lever will be equal to the magnitude of the force times distance d.

When using scalar quantities, the magnitude of the moment will be equal to the perpendicular distance between the line of action of the force and the point we are taking the moment about.

$$M = F * d \tag{3.1.1}$$

To determine the sign of the moment, we determine what type of rotation the force would cause. In this case, we can see that the force would cause the lever to rotate counterclockwise about point A. Counterclockwise rotations are caused by positive moments while clockwise rotations are caused by negative moments.

Another important factor to remember is that the value d is the perpendicular distance from the force to the point we are taking the moment about. We could measure the distance from point A to the head of the force vector, or the tail of the force vector, or really any point along the line of action of force F. The distance we need to use for the scalar moment calculation, however, is the shortest distance between the point and the line of action of the force. This will always be a line perpendicular to the line of action of the force, going to the point about which we are taking the moment.

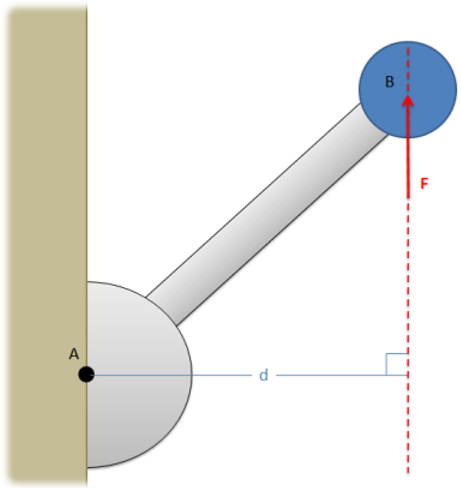


Figure 3.1.3: Distance d always needs to be the shortest length between the line of action of the force and the point we are taking the moment about. This distance will be perpendicular to the line of action of the force.

The Scalar Model in 3 Dimensions

For three-dimensional scalar calculations, we will still find the magnitude of the moment in the same way, multiplying the magnitude of the force by the perpendicular distance between the point and the line of action of the force. This perpendicular distance again is the minimum distance between the point and the line of action of the force. In some cases, finding this distance may be very difficult.

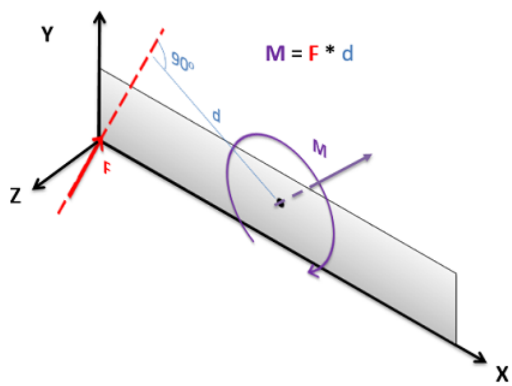


Figure 3.1.4: For moments in three dimensions, the moment vector will always be perpendicular to both the force vector $\vec{F}$ and the distance vector $\vec{d}$.

Another difficult factor in three dimensional scalar problems is finding the axis of rotation, as this is now more complex that just "clockwise or counterclockwise". The axis of rotation will be a line traveling though the point about which we are taking the moment, and perpendicular to both the force vector and the perpendicular displacement vector (the vector going from the point about which the moment was taken to the point of application of the force). While this is possible in any situation, it becomes very difficult if the force or displacement vectors do not lie in one of the three coordinate directions.

To further find the direction of the moment vector (which will act along the established line for axis of rotation), we will use the right-hand rule in a modified form. Wrap the fingers of your right hand around the axis of rotation line with your fingertips curling in the direction the body would rotate. If you do this, your thumb should point out along the line in the direction of the moment vector. This is an important last step, because we can rotate clockwise or counterclockwise in about any given axis of rotation. With the final moment vector, we known not only the axis of rotation, but which way the body would rotate about that axis.

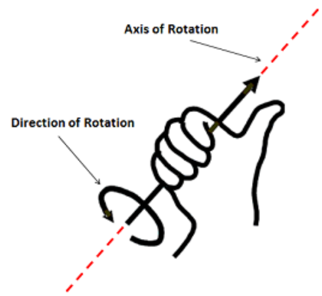


Figure 3.1.5: To use the right-hand rule, align your right hand as shown so that your thumb lines up with the axis of rotation for the moment and your curled fingers point in the direction of rotation for your moment. If you do this, your thumb will be pointing in the direction of the moment vector. Adapted from Public Domain image by Schorschi2.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/15h8bIQDjGE.

? Example 3.1.1

What is the moment that Force A exerts about point A? What is the moment that Force B exerts about Point A?

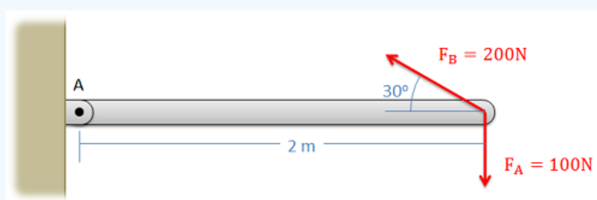


Figure 3.1.6: problem diagram for Example 3.1.1; a lever is attached to a wall with two forces exerted upon the lever's free end.

Solution

Video 3.1.7: Worked solution to example problem 3.1.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/E9Xq1fXdcyE.

? Example 3.1.2

What is the moment that this force exerts about point A? What is the moment this force exerts about point B?

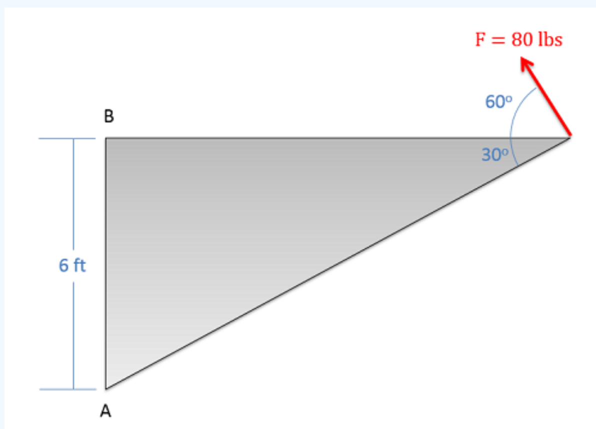


Figure 3.1.2: problem diagram for Example 3.1.2; a force is exerted upon one corner of a right triangle.

Solution

Video 3.1.3: Worked solution to example problem 3.1.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/9nb2q3EN5gs.

Example 3.1.3

What are the moments that each of the three tension forces exert about point A (the point where the beams come together)?

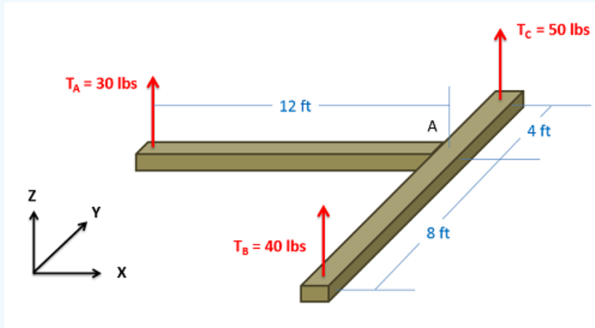


Figure 3.1.8: problem diagram for Example 3.1.3; three tension forces are exerted on the free ends of two beams that lie perpendicular to each other.

Solution

Video 3.1.4: Worked solution to example problem 3.1.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/2W9-K2KsTMU.

3.2: Varignon's Theorem

Varignon's Theorem, also often called the **principle of moments**, is a very useful tool in scalar moment calculations. In cases where the perpendicular distance is hard to determine, Varignon's Theorem offers an alternative to finding that distance.

In its basic form, Varignon's Theorem states that if we have two or more **concurrent forces**, the sum of the moments that each force creates about a single point will be equal to the moment created by the sum of those forces about the same point.

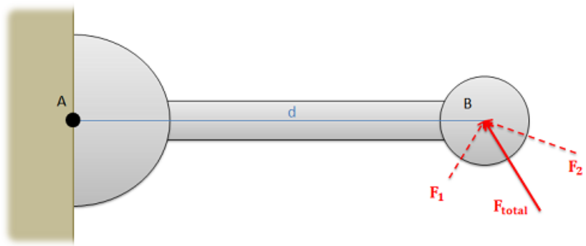


Figure 3.2.1: If the sum of $\vec{F}_1$ and $\vec{F}_2$ is $\vec{F}_{total}$, then we can assume that the sum of the moments about point A exerted by $\vec{F}_1$ and $\vec{F}_2$ will be equal to the moment exerted about point A by $\vec{F}_{total}$.

On its surface this doesn't seem that useful, but in practice we will often use this theorem in reverse by breaking down a force into components (the components being a set of concurrent forces). We can solve for the moment exerted by each component (where perpendicular distance d is easier to find) and then simply add together the moments from each component to find the moment from the original force.

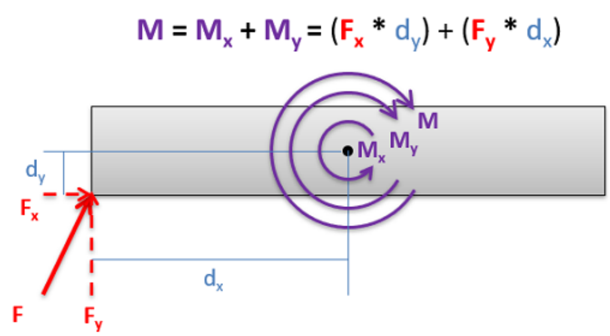


Figure 3.2.2: When finding the moment of force $\vec{F}$ about the center point, it will be easier to break down the force into components and find the moments of each component rather than trying to find the perpendicular distance directly using complex geometric relationships.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/XcxXyPv7Wp4.

Example 3.2.1

Use Varignon's Theorem to find the moment that the forces in the diagram below exert about point A.

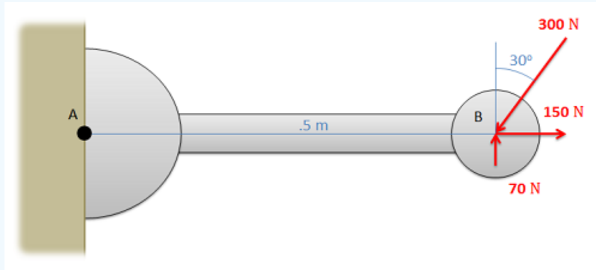


Figure 3.2.3: problem diagram for Example 3.2.1. A lever is attached to a wall at one end, and 3 different forces are applied to the lever's free end, 0.5 m from the base's point of contact with the wall (point A).

Solution

Video 3.2.2: Worked solution to example problem 3.2.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/JIpgZGxJWu4.

Example 3.2.2

Use Varignon's Theorem to find the moment that the force in the diagram below exerts about point B.

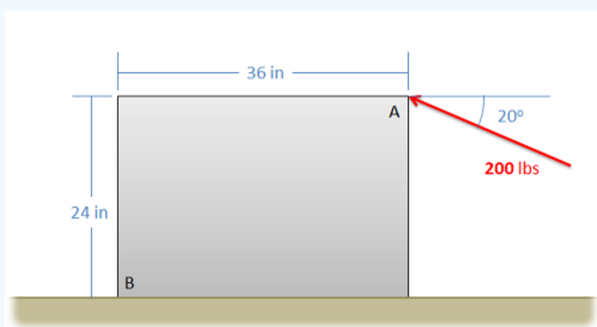


Figure 3.2.4: problem diagram for Example 3.2.2. A rectangular box, which can be considered a two-dimensional shape, sits on a flat surface and experiences a force applied to its upper right corner.

Solution

Video 3.2.3: Worked solution to example problem 3.2.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/jpaLEprFndA.

3.3: Couples

A **couple** is a set of equal and opposite forces that exerts a net moment on an object but no net force. Because the couple exerts a net moment without exerting a net force, couples are also sometimes called **pure moments**.

Figure 3.3.1: The two equal and opposite forces exerted on this lug wrench are a couple. They exert a moment on the lug nut on this wheel without exerting any net force on the wheel. Adapted from image by Steffen Heinz Caronna CC-BY-SA 3.0.

The moment exerted by a couple also differs from the moment exerted by a single force in that it is independent of the location you are taking the moment about. In the example below we have a couple acting on a beam. Each force has a magnitude F and the distance between the two forces is d.

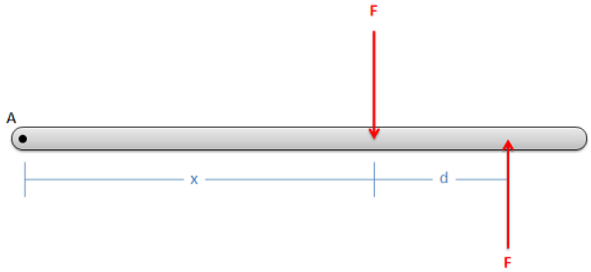


Figure 3.3.2: The moment exerted by this couple is independent of the of the distance x.

Now we have some point A, which is distance x from the first of the two forces. If we take the moment of each force about point A, and then add these moments together for the net moment about A we are left with the following formula.

$$M = -(F * x) + (F * (x + d)) \tag{3.3.1}$$

If we rearrange and simplify the formula above, we can see that the variable x actually disappears from the equation, leaving the net moment equal to the magnitude of the forces (F) times the distance between the two forces (d).

$$M = -(F * x) + (F * x) + (F * d) \tag{3.3.2}$$

$$M = (F * d) \tag{3.3.3}$$

This means that no matter what value of x we have, the magnitude of the moment exerted by the couple will be the same. The magnitude of the moment due to the couple is independent of the location we are taking the moment about. This will work in two or three dimensions as well. The magnitude of the moment due to a couple will always be equal to the magnitude of the forces times the perpendicular distance between the two forces.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/2U4APUz__Gk.

Example 3.3.1

What is the moment that the couple below exerts about point A?

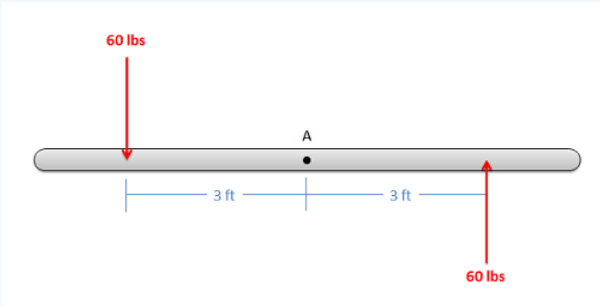


Figure 3.3.3: problem diagram for Example 3.3.1. A couple is applied to a rod, with each force being equidistant from point A on the rod.

Solution

Video 3.3.2: Worked solution to example problem 3.3.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/rfD-b6V5qNY.

Example 3.3.2

What is the moment that the couple below exerts about point A?

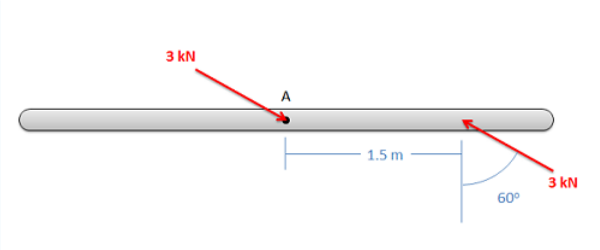


Figure 3.3.4: problem diagram for Example 3.3.2. A couple is applied to a rod, with one force applied directly at point A and the other being applied 1.5 meters to the right of A.

Solution

Video 3.3.3: Worked solution to example problem 3.3.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/s3-RJwKkH3Y.

3.4: Moment about a Point (Vector)

Given any point on an extended body, if there is a force acting on the body that does not travel through that point, then that force will cause a moment about that point. As discussed on the moments page, a moment is a force's tendency to cause rotation.

The Vector Method in 2 and 3 Dimensions

An alternative to calculating the moment via scalar quantities is to use the **vector method** or **cross product method**. For simple two-dimensional problems, using scalar quantities is usually sufficient, but for more complex problems the cross product method tends to be easier. The cross product method for calculating moments says that the moment vector of a force about a point will be equal to the cross product of a position vector $\vec{r}$, from the point to anywhere on the line of action of the force, and the force vector itself.

$$\vec{M} = \vec{r} \times \vec{F} \tag{3.4.1}$$

A big advantage of this method is that $\vec{r}$ does not have to be the shortest distance between the point and the line of action; it goes from the point to any part of the line of action. For any problem, there are many possible $\vec{r}$ vectors, but because of the way the cross product works, they should all result in the same moment vector in the end.

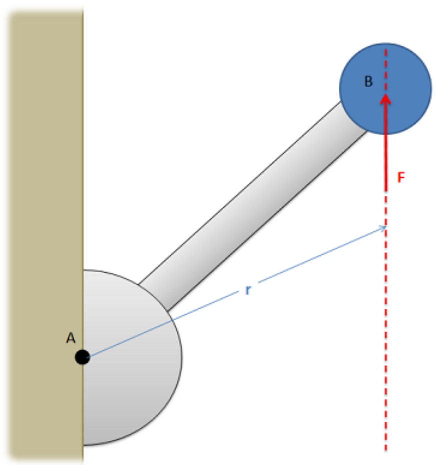


Figure 3.4.1: The moment vector of the force F about point A will be equal to the cross products of the r vector and the force vector. $\vec{r}$ is a vector from point A to any point along the line of action of the force.

It is important to note here that all quantities involved are vectors: $\vec{r}$, $\vec{F}$, and $\vec{M}$. Before you can solve for the cross product, you will need to write out $\vec{r}$ and $\vec{F}$ in vector component form. You will need to write out all three components of these vectors: for two-dimensional problems their z components will simply be zero, but those values are necessary for the calculations.

The moment vector you get will line up with the axis of rotation for the moment, where you can use the right-hand rule to determine if the moment is going clockwise or counterclockwise about that axis.

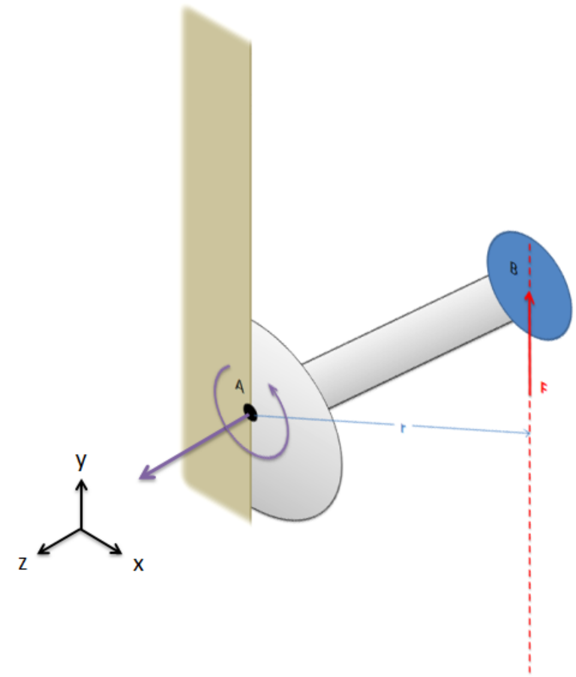


Figure 3.4.2: The result of $\vec{r} \times \vec{F}$ will give us the moment vector. For this two-dimensional problem, the moment vector is pointing in the positive z direction. We can use the right-hand rule to determine the direction of rotation from the moment: line our right thumb up with the moment vector and our curled fingers will point in the direction of rotation from the moment.

Finally, it is also important to note that taking the cross product, unlike multiplication, is not communicative. This means that the order of the vectors matters, and $\vec{r} \times \vec{F}$ will not be the same as $\vec{F} \times \vec{r}$. It is important to always use $\vec{r} \times \vec{F}$ when calculating moments.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/Jt2AE2yZFEQ.

? Example 3.4.1

What is the moment that this force exerts about point A? What is the moment this force exerts about point B?

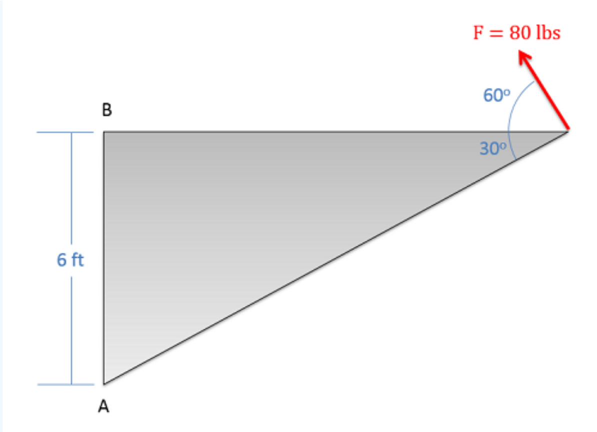


Figure 3.4.3: problem diagram for Example 3.4.1. A force is applied to one corner of a right triangle, producing moments about the triangle's other corners.

Solution

Video 3.4.2: Worked solution to example problem 3.4.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/wn1xJZMpDY4.

? Example 3.4.2

Determine the moment that the tension in the cable exerts about the base of the pole (leave the moment in vector form). What is the magnitude of the moment the tension exerts about this point?

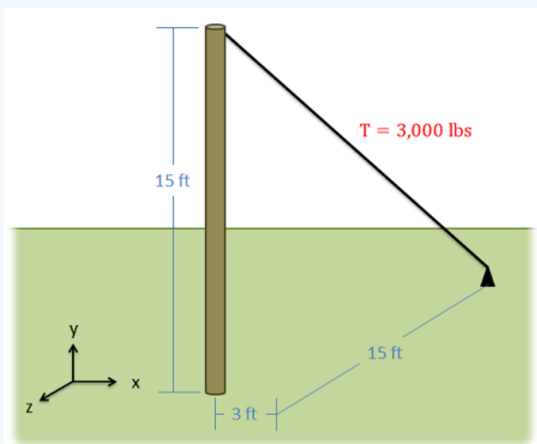


Figure 3.4.4: problem diagram for Example 3.4.2; a three-dimensional arrangement of an upright pole and a cable that connects the pole to the ground.

Solution

Video 3.4.3: Worked solution to example problem 3.4.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/ioqy98jZ0Vg.

3.5: Moment of a Force About an Axis

It is sometimes useful to be able to calculate the moment a force exerts about a certain axis that is relevant to the problem. An example would be a force on the vault door in the image below. If we took the moment about a point (such as one of the hinges on the door), we may find that the moment vector does not line up with the axis of this hinge. In that case, the component of the moment vector that lines up with the axis of the hinge will cause a rotation, while the component of the moment vector that does not line up with the axis of the hinge will cause reaction moments in the hinge. If we are only interested in the rotation of the door, we will want to find the moment that the force exerts specifically about the axis of the hinges.

Figure 3.5.1: The hinge on a door such as the one shown above will only allow for rotation along the axis of the hinge. Since this corresponds to moments along the axis of the hinge, it may be useful to specifically calculate the moment a force exerts about the axis of the hinge. Public Domain image, no author listed.

Calculating the Moment About an Axis via the Dot Product

To find the moment of a force about a specific axis, we find the moment that the force exerts about some point on that axis and then we find the component of the moment vector that lines up with the axis we are interested in.

To do this mathematically, we use the cross product to calculate the moment of the force about any point along the axis, and then we take the dot product of a unit vector $\vec{u}$ along the axis and the moment vector we just calculated.

$$M = \vec{u} \cdot (\vec{r} \times \vec{F}) \tag{3.5.1}$$

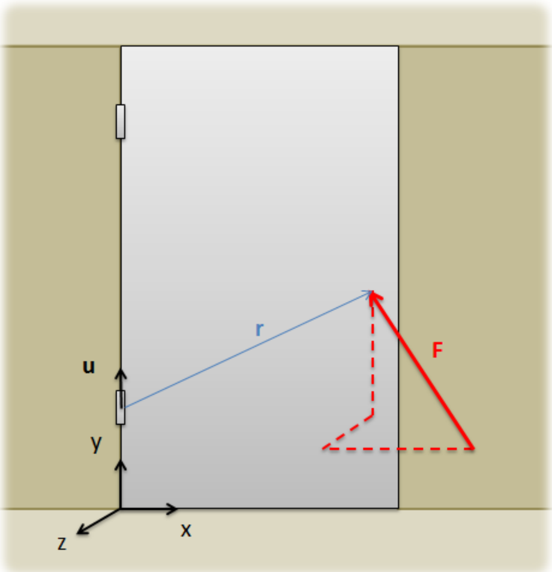


Figure 3.5.2: The moment of a force about an axis is the dot product of $\vec{u}$ and the cross product of $\vec{r}$ and $\vec{F}$.

The unit vector $\vec{u}$ has a magnitude of one and will be pointing in the direction of the axis we are interested in. Your final answer from this operation will be a scalar value (having a magnitude but no direction). This is the magnitude of the moment about the given axis, with the direction being specified by the unit vector $\vec{u}$.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/FMu8XpvBAMo.

Example 3.5.1

A flat L-shaped plate is attached via two hinges as shown in the diagram below. If force F acts on the plate as shown in the diagram, what is the moment that force F exerts about the axis of the hinges?

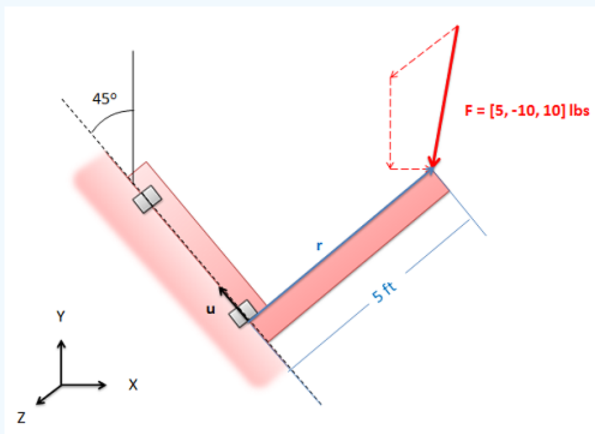


Figure 3.5.3: problem diagram for Example 3.5.1; a two-dimensional piece of metal experiences a force with components in all three dimensions that causes the metal piece to rotate on hinges.

Solution

Video 3.5.2: Worked solution to example problem 3.5.1, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/bP9iKRCCrAc.

3.6: Equilibrium Analysis for a Rigid Body

For an **rigid body** in static equilibrium—that is, a non-deformable body where forces are not concurrent—the sum of both the **forces** and the **moments** acting on the body must be equal to zero. The addition of moments (as opposed to particles, where we only looked at the forces) adds another set of possible equilibrium equations, allowing us to solve for more unknowns as compared to particle problems.

Moments, like forces, are vectors. This means that our vector equation needs to be broken down into scalar components before we can solve the equilibrium equations. In a two-dimensional problem, the body can only have clockwise or counterclockwise rotation (corresponding to rotations about the z axis). This means that a rigid body in a two-dimensional problem has three possible equilibrium equations; that is, the sum of force components in the x and y directions, and the moments about the z axis. The sum of each of these will be equal to zero.

For a two-dimensional problem, we break our one vector force equation into two scalar component equations.

$$\sum \vec{F} = 0 \tag{3.6.1}$$

$$\sum F_x = 0\,; \; \sum F_y = 0 \tag{3.6.2}$$

The one moment vector equation becomes a single moment scalar equation.

$$\sum \vec{M} = 0 \tag{3.6.3}$$

$$\sum M_z = 0 \tag{3.6.4}$$

If we look at a three-dimensional problem we will increase the number of possible equilibrium equations to six. There are three equilibrium equations for force, where the sum of the components in the x, y, and z directions must be equal to zero. The body may also have moments about each of the three axes. The second set of three equilibrium equations states that the sum of the moment components about the x, y, and z axes must also be equal to zero.

We break the forces into three component equations.

$$\sum \vec{F} = 0 \tag{3.6.5}$$

$$\sum F_x = 0\,; \; \sum F_y = 0\,; \; \sum F_z = 0 \tag{3.6.6}$$

Then we also break the moments into three component equations.

$$\sum \vec{M} = 0 \tag{3.6.7}$$

$$\sum M_x = 0\,; \; \sum M_y = 0\,; \; \sum M_z = 0 \tag{3.6.8}$$

Finding the Equilibrium Equations

As with particles, the first step in finding the equilibrium equations is to draw a free body diagram of the body being analyzed. This diagram should show all the force vectors acting on the body. In the free body diagram, provide values for any of the known magnitudes, directions, and points of application for the force vectors and provide variable names for any unknowns (either magnitudes, directions, or distances).

Next you will need to choose the x, y, and z axes. These axes do need to be perpendicular to one another, but they do not necessarily have to be horizontal or vertical. If you choose coordinate axes that line up with some of your force vectors you will simplify later analysis.

Once you have chosen axes, you need to break down all of the force vectors into components along the x, y and z directions (see the vectors page in Appendix 1 page for more details on this process). Your first equation will be the sum of the magnitudes of the components in the x direction being equal to zero, the second equation will be the sum of the magnitudes of the components in the y direction being equal to zero, and the third (if you have a 3D problem) will be the sum of the magnitudes in the z direction being equal to zero.

Next you will need to come up with the the moment equations. To do this you will need to choose a point to take the moments about. Any point should work, but it is usually advantageous to choose a point that will decrease the number of unknowns in the equation. Remember that any force vector that travels through a given point will exert no moment about that point. To write out the moment equations, simply sum the moments exerted by each force (adding in pure moments shown in the diagram) about the given point and the given axis, and set that sum equal to zero. All moments will be about the z axis for two-dimensional problems, though moments can be about the x, y and z axes for three-dimensional problems.

Once you have your equilibrium equations, you can solve these formulas for unknowns. The number of unknowns that you will be able to solve for will again be the number of equations that you have.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/OiJ2xbMIixY.

Example 3.6.1

The car below has a weight of 1500 lbs with the center of mass 4 ft behind the front wheels of the car. What are the normal forces on the front and the back wheels of the car?

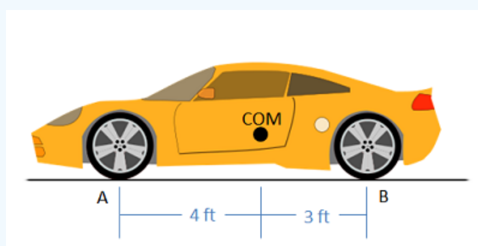


Figure 3.6.1: problem diagram for Example 3.6.1. Adapted from the public domain image by Ebaychatter0.

Solution

Video 3.6.2: Worked solution to example problem 3.6.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/1LD5QW-70PA.

Example 3.6.2

A 5-meter-long beam has a fixed connection to a wall at point A and a force acting as shown at point B. What are the reaction forces acting on the beam at point A?

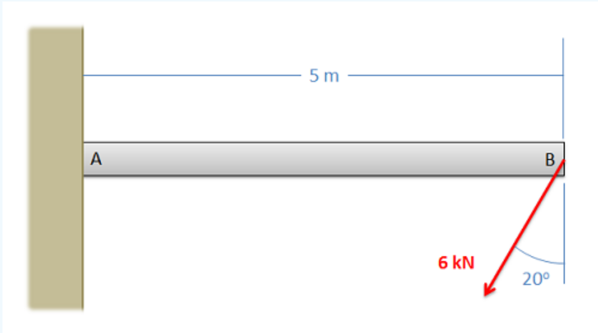


Figure 3.6.2: problem diagram for Example 3.6.2. A horizontal beam attached to the wall at one end experiences a force applied at its free end.

Solution

Video 3.6.3: Worked solution to example problem 3.6.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/JrVV7k1aQEk.

Example 3.6.3

A ladder with negligible mass is supporting a 120-lb person as shown below. If the contact point at A is frictionless, and the contact point at B is a rough connection, determine the forces acting at contact points A and B.

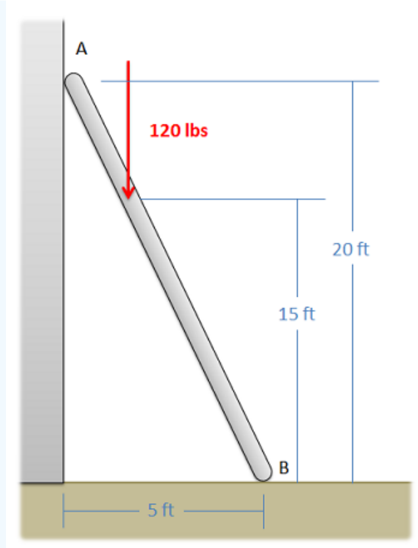


Figure 3.6.3: problem diagram for Example 3.6.3. A ladder with its base on a rough floor leans against a frictionless wall, with a person standing on the ladder partway up.

Solution

Video 3.6.4: Worked solution to example problem 3.6.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/WzkAnPdhao4.

? Example 3.6.4

Member ABC is 6 meters long, with point B being at its midpoint. Determine all forces acting on member ABC.

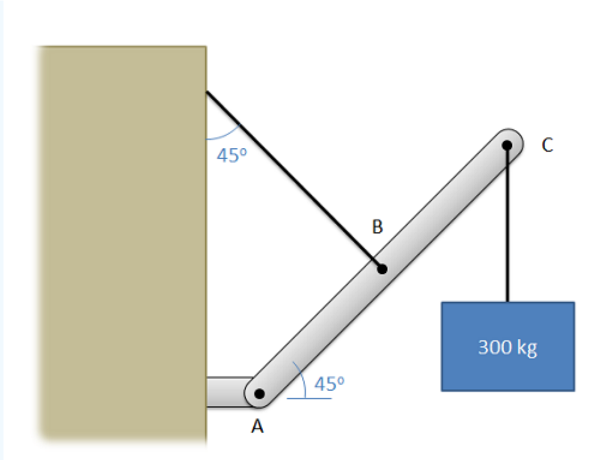


Figure 3.6.4: problem diagram for Example 3.6.4. A diagonal structural member is attached to the wall at one end, connected to the wall via cable at its midpoint, and holding up a 300-kg load at its free end.

Solution

Video 3.6.5: Worked solution to example problem 3.6.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/sMQrjwUMpSQ.

Example 3.6.5

While sitting in a chair, a person exerts the forces in the diagram below. Determine all forces acting on the chair at points A and B. (Assume A is frictionless and B is a rough surface).

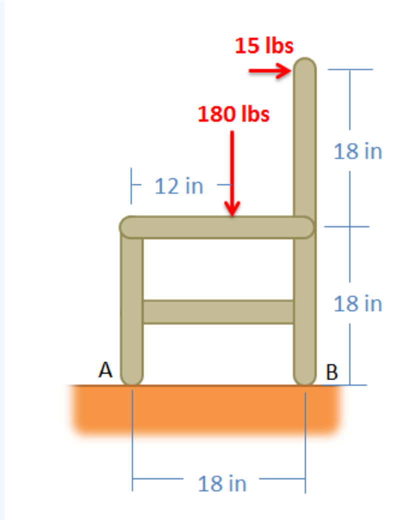


Figure 3.6.5: problem diagram for Example 3.6.5. A chair is placed on a flat surface, with that surface assumed to be frictionless where it contacts the chair's front leg (point A) and exerting friction where it contacts the back leg (point B).

Solution

Video 3.6.6: Worked solution to example problem 3.6.5, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/nSOxK1ZMggA.

? Example 3.6.6

The trailer shown below consists of a deck with a weight of 250 lbs on an axle with wheels with a weight of 350 lbs. Assume the weight forces act in the center of each component. If we wish the tongue weight (F_T) of the unloaded trailer to be 50 lbs, what is the distance d from the front where we must place the axle?

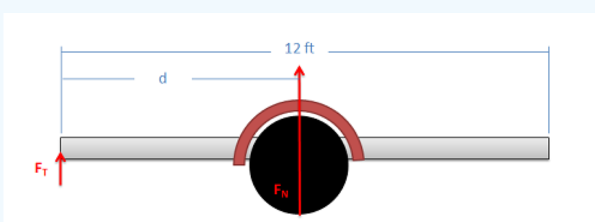


Figure 3.6.6: problem diagram for Example 3.6.6. A trailer consists of a flat rectangular deck on top of two wheels on an axle.

Solution

Video 3.6.7: Worked solution to example problem 3.6.6, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/wpEBuitLD5s.

Example 3.6.7

A 12-inch-by-24-inch flat steel sign is supported by two cables, each 6 inches from the edge of the sign. The sign has a weight of 10 lbs, and the wind is causing the sign to sit at an angle of 10 degrees from vertical (the y axis). If we treat the wind as a point force acting in the negative z direction on the center of the sign, how strong must the wind force be to cause this ten-degree angle?

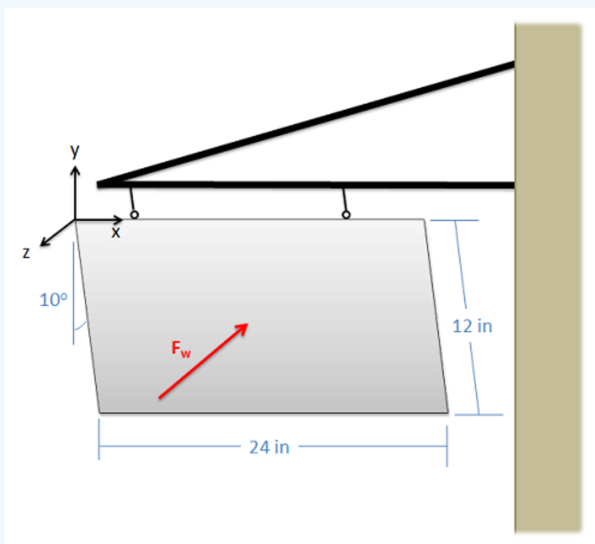


Figure 3.6.7: problem diagram for Example 3.6.7. A hanging sign experiences a wind whose direction points into the screen, causing the sign to make a 10° angle with the plane of the screen.

Solution

Video 3.6.8: Worked solution to example problem 3.6.7, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/pR-0xbj8wF0.

Example 3.6.8

A sixty-kilogram acoustic panel is suspended by three cables as shown below. Assuming the panel has a uniformly distributed weight, what is the tension in each of the cables?

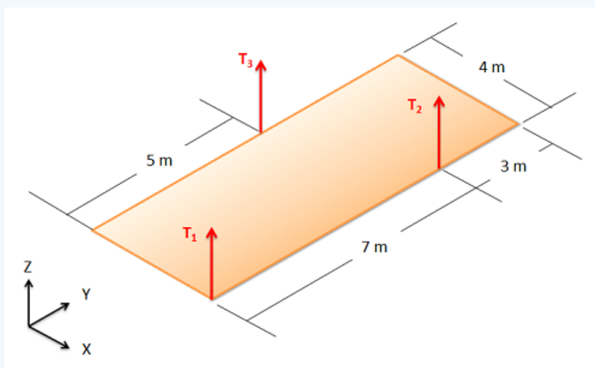


Figure 3.6.8: problem diagram for Example 3.6.8. A uniform rectangular panel is suspended from above by 3 cables located at different points along its edges.

Solution

Video 3.6.9: Worked solution to example problem 3.6.8, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Kbsc1m0f9pQ.

3.7: Chapter 3 Homework Problems

Exercise 3.7.1

An 18-inch shelf is supported by a pin joint at point A, and a cable at point B. The shelf itself has a weight of 60 lbs. If we want the net moment about point A to be zero, what should the tension in the cable be?

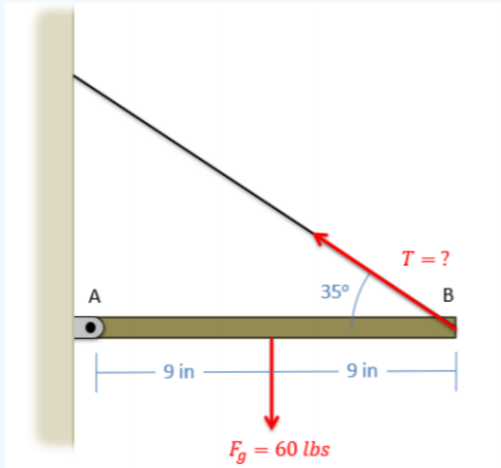


Figure 3.7.1: problem diagram for Exercise 3.7.1; a horizontal shelf is attached to a wall with a pin joint at one end and a cable at the other end.

Solution

$$T = 52.30\ lbs$$

Exercise 3.7.2

What is the moment the force shown below exerts about Point A? About Point B?

Hint: use Varginon's Theorem.

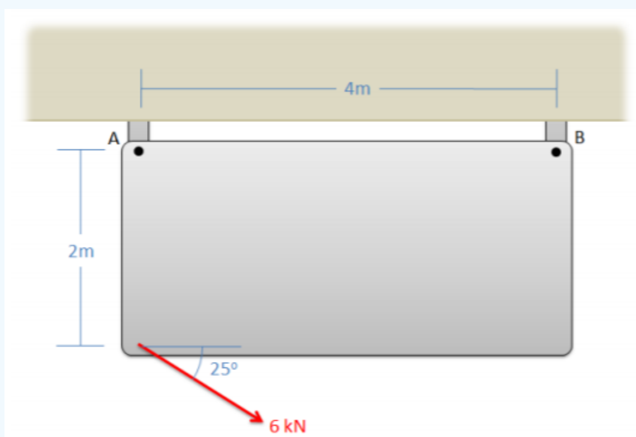


Figure 3.7.2: problem diagram for Exercises 3.7.2; a rectangular slab is bolted to a wall at two points, A and B, with a force is exerted on one of the rectangle's unattached corners.

Solution

$$M_A = 10.88\ kNm$$

$$M_B = 21.02\ kNm$$

Exercise 3.7.3

You are attempting to rotate a heavy table about the base of one leg at point O and are going to exert a 100-lb force at the opposite end. Person A recommends pulling straight up, while person B recommends pulling up at 30° from vertical. What would be the moment about point O, in inch-pounds, in either case?

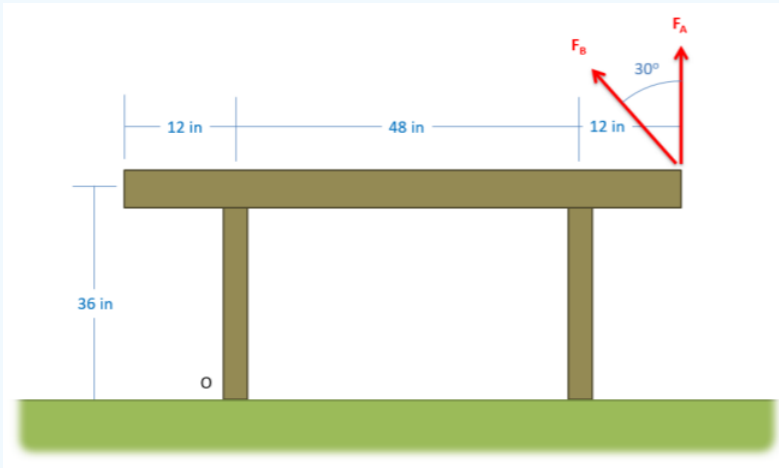


Figure 3.7.3: problem diagram for Exercise 3.7.3; a table is rotated about the base of one leg, with one of the two proposed forces applied at the tabletop at the opposite end.

Solution

$$M_{AO} = 6000 \text{ in-lbs}$$

$$M_{BO} = 6996.15 \text{ in-lbs}$$

Exercise 3.7.4

Determine the moment vectors that each of the three tension forces in the diagram below exerts about the origin point O. Provide the answers in vector form with units.

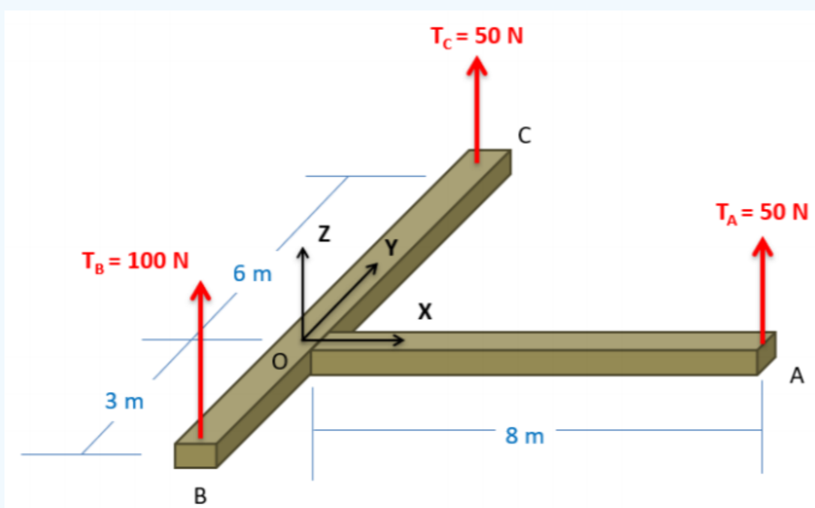


Figure 3.7.4: problem diagram for Exercise 3.7.4; two straight beams lying in the xy-plane, joined perpendicularly to each other, experience 3 upward tension forces (in the $+z$ direction).

Solution

$$M_{AO} = [0,\ -400,\ 0]\ Nm$$

$$M_{BO} = [-300,\ 0,\ 0]\ Nm$$

$M_{CO} = [300,\ 0,\ 0] Nm$

? Exercise 3.7.5

You exert a 50-lb force on the side of a fridge as shown below. Assuming the fridge is sitting on a rough surface and not moving, what is the magnitude of the moment exerted by the couple consisting of the pushing force and the friction force?

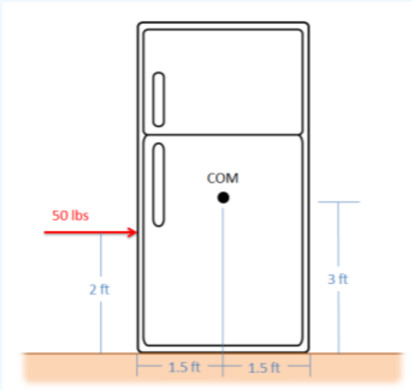


Figure 3.7.5: problem diagram for Exercise 3.7.5; a force is exerted on a refrigerator that sits on a flat surface, without causing it to move.

Solution

$M = -100$ft-lbs

? Exercise 3.7.6

A space station consists of a large ring that spins in order to provide an artificial gravity for the astronauts in the station. To start the station spinning, a pair of thrusters is attached to the outside of the ring, each pointing in opposite directions as shown below.

a) If we want to exert a 10 kN-m moment with the thrusters, and the ring has a diameter of 45 meters, what thrust force should each thruster produce?

b) If we were to use the same thrusters on a 60-meter diameter ring, what moment would they exert?

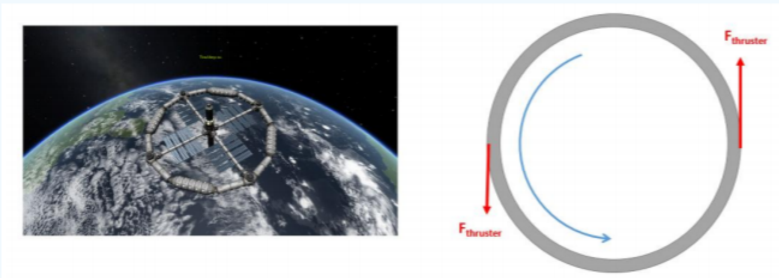


Figure 3.7.6: problem diagram for Exercise 3.7.6. An example of the spinning ring-shaped space station as described in the problem (left); a diagram of the locations and directions of the thruster forces on the ring (right).

Solution

a) $F_{thruster} = 222.22\ N$

b) $M_{thruster} = 13.33\ kNm$

❓ Exercise 3.7.7

A 60-N force is applied in the yz plane, 40° from the y direction, on an L-shaped bar as shown below.

a) What is the moment vector this force exerts about point O?

b) What is the overall magnitude of the moment about point O?

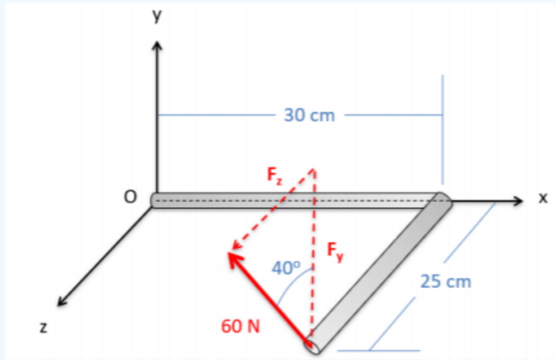


Figure 3.7.7: problem diagram for Exercise 3.7.7; an L-shaped bar lies in the xz plane with one end located at the origin, point O, and the other experiencing a force in the yz direction.

Solution

$$M_O = [-11.49,\ -11.57,\ 13.79]\ Nm$$

$$|M| = 21.36\ Nm$$

❓ Exercise 3.7.8

What is the moment that the force shown in the diagram exerts about point O? About the axis of the cylindrical shaft (the y-axis)?

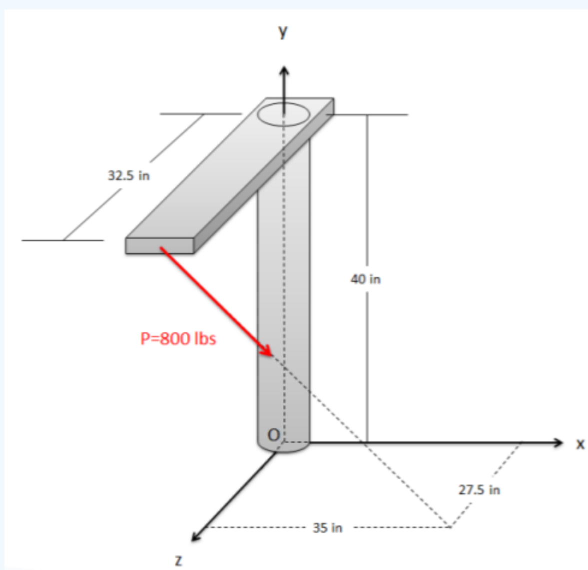


Figure 3.7.8: problem diagram for Exercise 3.7.8; an L-shaped part that extends in the y and z directions, with one end located at the origin O, experiences a force in the xz direction applied at the opposite end.

Solution

$$M_O = [16484,\ 17046,\ -20979]\ in-lbs$$

$$M_y = 17046\ in-lbs$$

Exercise 3.7.9

The diving board shown below is supported by a pin joint at A and frictionless support at B. A 150-lb diver is standing at the end of the board. Determine the reaction forces acting on the diving board at points A and B.

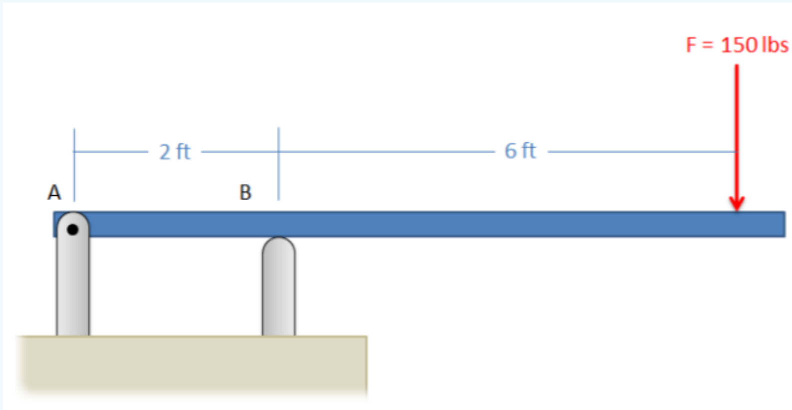


Figure 3.7.9: problem diagram for Exercise 3.7.9; a diving board is supported by a pin joint at its leftmost end (point A) and a frictionless support two feet to the right (point B), with a 150-lb diver standing 6 feet to the right of B.

Solution

$F_{AX} = 0$

$F_{AY} = -450\,lbs$

$F_{BY} = 600\,lbs$

Exercise 3.7.10

A simplified crane is shown lifting a 400-kg load. The crane is supported by a pin joint at A, and a cable at B. Assuming the crane arm is in equilibrium, what are the reaction forces at A and the tension at B?

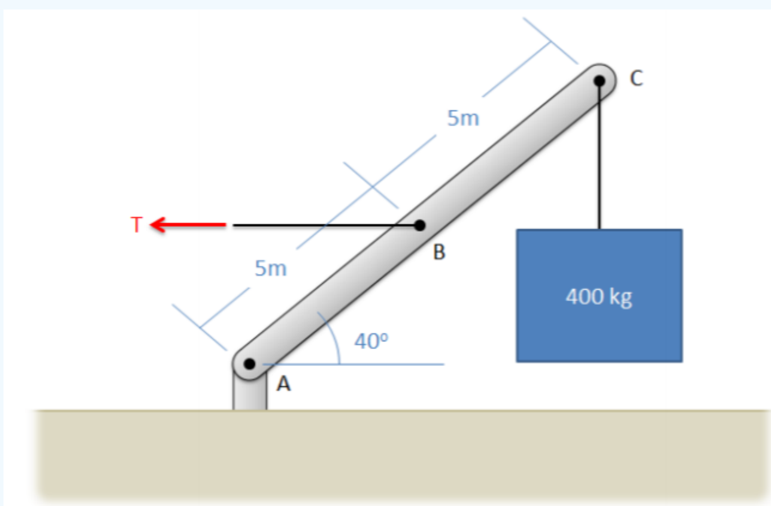


Figure 3.7.10: problem diagram for Exercise 3.7.10; a crane arm is attached to a vertical support at one end (A), is connected to a horizontal cable at its midpoint (B), and lifts at a 400-kg load at the other end.

Solution

$F_{AX} = 9352.9\,N$

$F_{AY} = 3924\,N$

$T_B = 9352.9\,N$

Exercise 3.7.11

An 8-foot ladder sits propped up against a wall at a 60-degree angle as shown below. It has a weight of 50 lbs acting at its center point and supports a 120-lb woman 6 feet from the bottom. Assume that friction acts at the bottom of the ladder, but not the top. What are the normal forces acting at the bottom and top of the ladder, and what is the friction force acting at the bottom of the ladder?

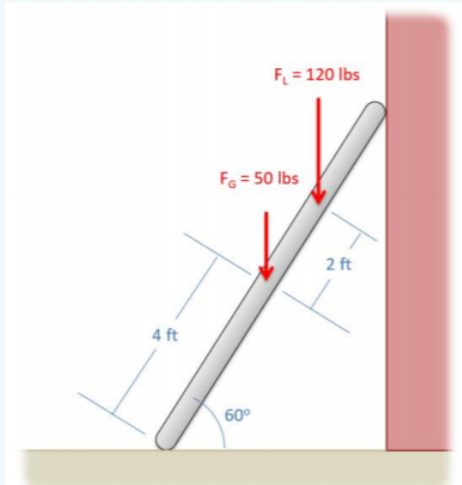


Figure 3.7.11: problem diagram for Exercise 3.7.11; an 8-ft ladder weighing 50 lbs and supporting a 120-lb woman who has climbed 75% of the way up leans, at 60° above the horizontal, against a wall.

Solution

$F_{N\ Top} = 66.4\ lbs$

$F_{N\ Bottom} = 170\ lbs$

$F_f = 66.4 lbs$

Exercise 3.7.12

An SUV with a weight of 4200 lbs and a center of mass located as shown below is parked pointed downhill on a 10-degree incline. The parking is engaged, locking up the back wheels but not the front wheels. What is the expected normal force at the front wheels, the expected normal force at the back wheels, and the expected friction force at the back wheels assuming the SUV does not slip?

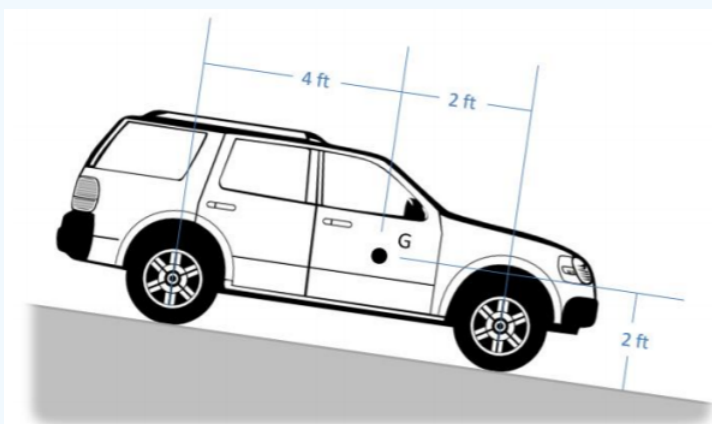


Figure 3.7.12: problem diagram for Exercise 3.7.12; an SUV with front and rear wheels 6 feet apart, and a center of mass 2 feet above the ground and 2 feet behind the front wheel, is parked pointing downhill on a 10° incline.

Solution

$F_f = 729.3\, lbs$

$F_{N\ front} = 3000.6\, lbs$

$F_{N\ back} = 1135.6\, lbs$

Exercise 3.7.13

A cart with a mass of 3500 kg sits on an inclined surface as shown below. Determine the reaction forces acting on each wheel of the cart as well as the tension in the cable supporting the cart.

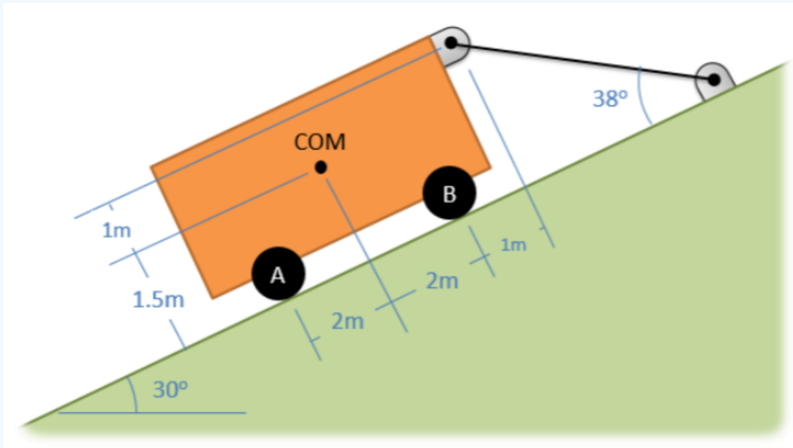


Figure 3.7.13: a two-wheeled cart is parked facing uphill on a 30° slope. A cable stretches from the front of the cart to a support on the incline, making a 38° angle with the plane of the incline.

Solution

$T = 21786\, N$

$F_A = 7222\, N;\ F_B = 35925\, N$

Exercise 3.7.14

The lighting rig above a stage consists of two 100-lb, uniform beams joined together in a T as shown below (assume the weight acts in the center of each beam). The rig is supported by three cables at A, B, and C. Determine the tension in each of the three cables.

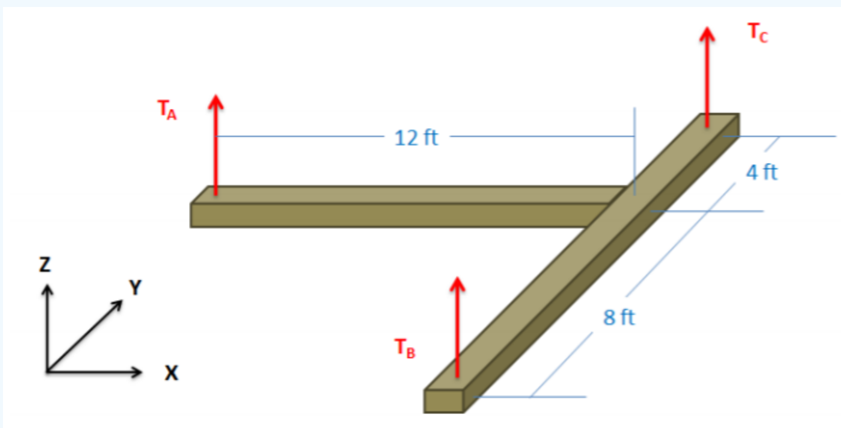


Figure 3.7.14: problem diagram for Exercise 3.7.14; two beams attached perpendicularly to each other lie in the xy plane in a T shape experience upward tension forces from 3 cables, one attached to each free end of the T.

Solution

$T_A = 50\, lbs;\ T_B = 66.7\, lbs;\ T_C = 83.3\, lbs$

Exercise 3.7.15

A 9-meter-long pole with a mass of 100 kg is suspended horizontally, 4 meters from the ceiling with three cables as shown below. Assuming the center of mass of the pole is at the center point of the pole, what is the expected tension in each of the three cables?

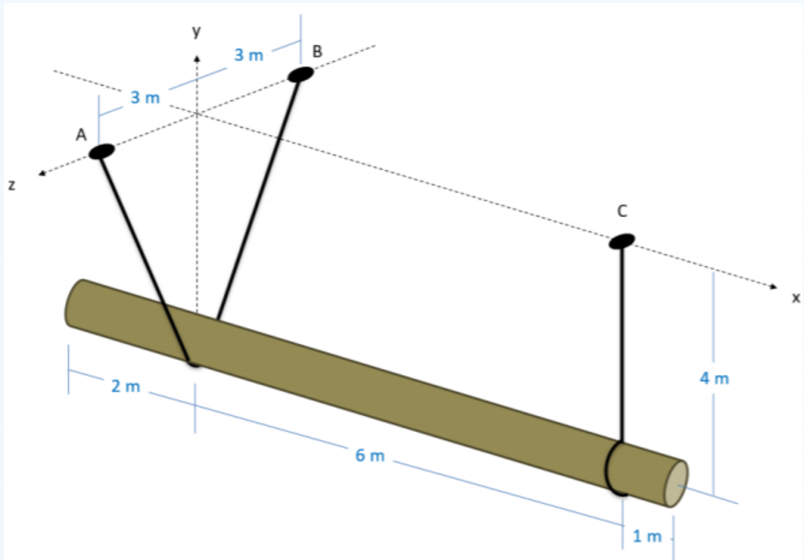


Figure 3.7.15: problem diagram for Exercise 3.7.15; a pole hangs 4 meters below the ceiling, suspended by 3 cables attached to the pole at different locations and angles.

Solution

$T_A = 357.66\, N;\ T_B = 357.66\, N;\ T_C = 408.75\, N$

CHAPTER OVERVIEW

4: Statically Equivalent Systems

4.0: Video Introduction to Chapter 4

Video introduction to the topics covered in Chapter 4, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/jXY48L2sRR0.

4.1: Statically Equivalent Systems

Two sets of forces are considered **statically equivalent** if they **cause the same set of reaction forces** on a body. Because this is true, the two statically equivalent sets of forces are **interchangeable** in statics analysis.

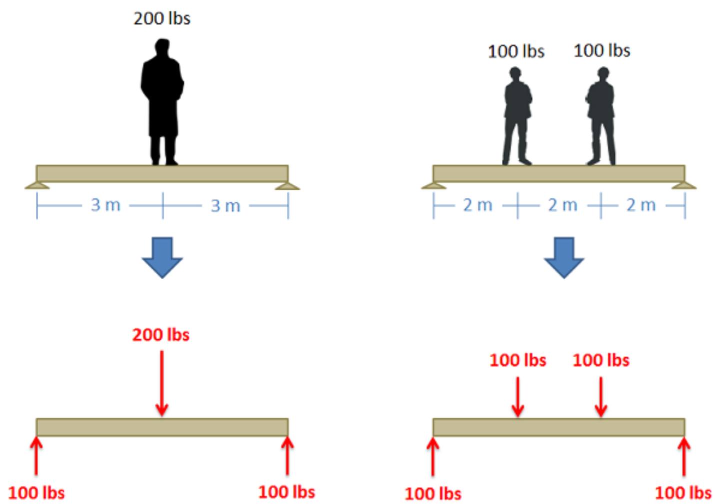


Figure 4.1.1: A single 200-lb man standing at the center of a beam causes the same reaction forces as two 100-lb children standing evenly along the beam as shown above. The weight force of the man is therefore statically equivalent to the weight forces of the two children.

Determining if Forces are Statically Equivalent:

To determine if two sets of forces are statically equivalent, you must solve for the reaction forces in both cases. If the reaction forces are the same then the two sets of forces must be statically equivalent. For any one set of forces, there are an infinite number of sets of forces that are statically equivalent to original set of forces.

Finding a Single Equivalent Point Force:

In statics analysis, we are usually looking to simplify a problem by turning multiple forces into a single, statically equivalent force. To find a single point force that is equivalent to multiple point forces you can use the following procedure.

1. Solve for the reaction forces in the original scenario.
2. Draw a new free body diagram with these reaction forces. You will also add one force with an unknown magnitude, direction, and point of application to your diagram. This is the single point load that will be equivalent to your original set of forces.
3. Write out the equations of equilibrium for this scenario, including the known values for the reaction forces.
4. First, solve the force equations to find the x and y components of this unknown force (or x, y and z components for a 3D problem). This can be used to find the magnitude and direction of the statically equivalent point force.
5. Next, use the moment equation (or equations, for 3D problems) to determine the location of the statically equivalent point force.

Video 4.1.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/r0pwzQ_Ge00.

Example 4.1.1

Determine if the two sets of forces shown below are statically equivalent.

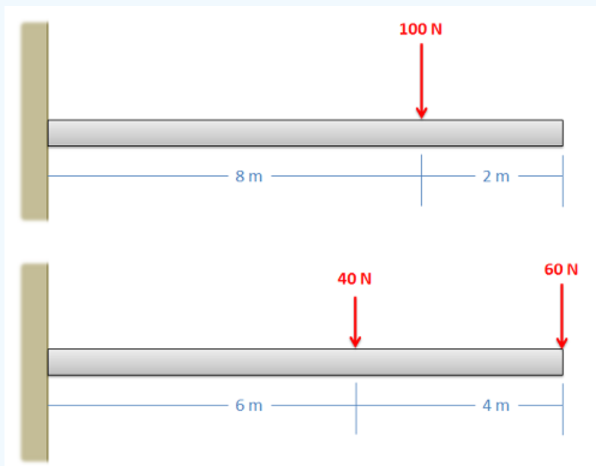


Figure 4.1.2: problem diagram for Example 4.1.1; two identical bars experience the same total magnitude and direction of external force, applied in different locations and distributions.

Solution

Video 4.1.2: worked solution to example problem 4.1.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Xv5DyvQD29E.

4.2: Resolution of a Force into a Force and a Couple

As discussed on the sections on moments, a force can have a tendency to cause both a linear and angular acceleration. For example, below is diagram of a force acting on an extended body. If we were to think about everything relative to the center of mass of this body, there would be some force acting on the center of mass that would cause the same linear acceleration and some pure moment (couple) that would cause the same angular acceleration. This would be a force and couple that is **statically equivalent** to the original force, though the point of application of the force changes.

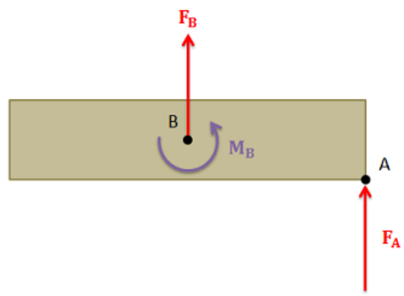


Figure 4.2.1: The original force acting at A would cause both linear and angular acceleration. The force at B would cause the same linear acceleration and the moment at B would cause the same angular acceleration. The force and the moment at B are statically equivalent to the original force at A.

The process of transforming one force applied at one point, into a force and a couple at some other point is known as **resolving a force into a force and a couple**. There are a few reasons that we may want to do this, but one primary reason is to find the **equivalent force couple system** for a complex set of forces and moments. The equivalent force couple system is used to simplify more complex analysis, and consists of a single force and a single pure moment (couple) that are statically equivalent to some more complex combination of forces and moments. An important first step in finding the equivalent force couple system is to resolve all the forces so that everything is acting at the same point.

In order to visualize the process of resolving a force into a force and a couple, you can use the process outlined in the diagram below. Imagine we have a body with a force acting at some point A. We want to resolve the force into a force and a couple about some other point B. To do this we will first add two forces to the diagram at point B. One will have the same magnitude and direction as the original force and the other will be equal and opposite to the original force. Because these two forces are equal, opposite, and collinear, this will not change the situation (it's the equivalent to adding zero to an equation). Now, with these three forces acting on the diagram, we can break it down into two sets. The first is a force acting at point B with the same magnitude and direction as the original force. The other two forces act as a couple, exerting a pure moment about point B. Finally, we can redraw the system as a force acting at point B and the pure moment acting about point B.

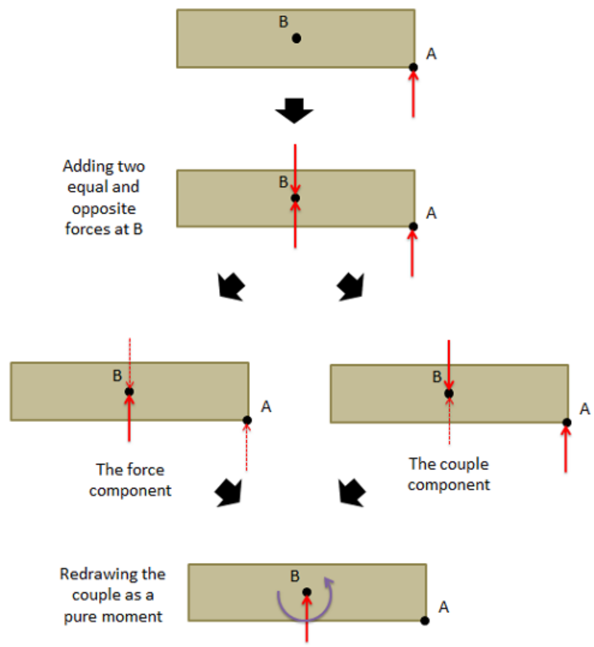


Figure 4.2.2: The process of resolving a force about some point A into a force and a couple about some point B.

As a shortcut to the process described above, we can also see that the force in the equivalent force couple system will always have a magnitude and direction equal to the original force and the couple will be equal to the moment exerted by the original force about the new point of application.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/sLixBOWBCuY.

Example 4.2.1

Resolve the force shown below to a force and a couple acting at point A.

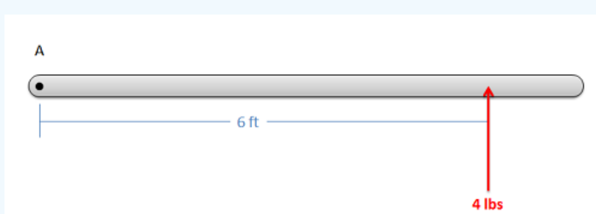


Figure 4.2.3: problem diagram for Example 4.2.1; a rod experiences a single, upwards force applied 6 feet from the left end of the rod (point A).

Solution

Video 4.2.2: Worked solution to example problem 4.2.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Wfl3S_5FD0Q.

Example 4.2.2

Resolve the force shown below to a force and a couple acting at point A.

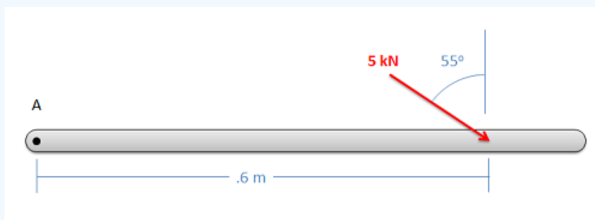


Figure 4.2.4: problem diagram for Example 4.2.2; a rod experiences a single force, pointing downwards and to the right, applied 0.6 meters from the left end of the rod (point A).

Solution

Video 4.2.3: Worked solution to example problem 4.2.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/1hRrKhWzf98.

4.3: Equivalent Force Couple System

Every set of forces and moments has an **equivalent force couple system**. This is a single force and pure moment (couple) acting at a single point that is **statically equivalent** to the original set of forces and moments.

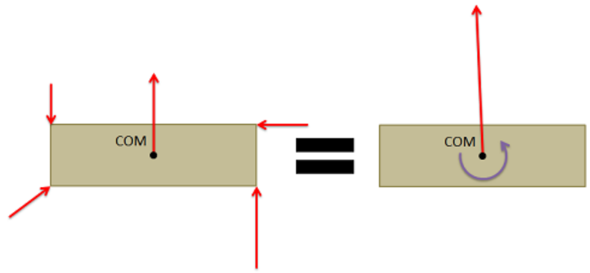


Figure 4.3.1: Any set of forces on a body can be replaced by a single force and a single couple acting that is statically equivalent to the original set of forces and moments. This set of an equivalent force and a couple is known as the equivalent force couple system.

To find the equivalent force couple system, you simply need to follow the steps below.

1. First, choose a point to take the equivalent force couple system about. Any point will work, but the point you choose will affect the final values you find for the equivalent force couple system. Traditionally, this point will either be the center of mass of the body or some connection point for the body.
2. Next, resolve all the forces not acting though that point to a force and a couple acting at the point you chose.
3. To find the "force" part of the equivalent force couple system, add together all the force vectors. This will give you the magnitude and the direction of the force in the equivalent force couple system.
4. To find the "couple" part of the equivalent force couple system, add together any moment vectors (this could be moments originally acting on the body, or moments from the resolution of the forces into forces and couples). This will give you the magnitude and direction of the pure moment (couple) in the equivalent force couple system.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/bs6Tnlje3IU.

Example 4.3.1

Find the equivalent force couple system for the forces shown below about point A.

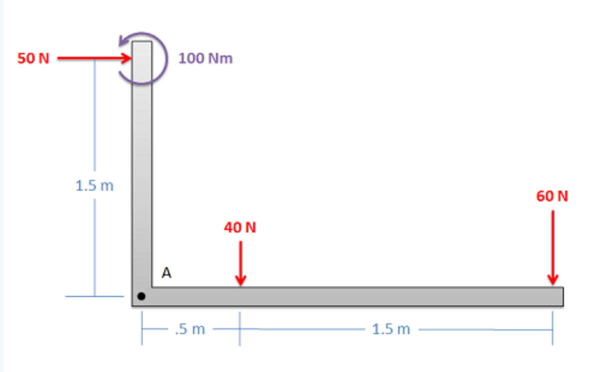


Figure 4.3.2: problem diagram for Example 4.3.1. An L-shaped part whose two arms connect at point A experiences a rightwards force and a counterclockwise moment about the end of its vertical arm, as well as two downwards forces along its horizontal arm.

Solution

Video 4.3.2: Worked solution to example problem 4.3.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/cw-rwBZef5w.

4.4: Distributed Forces

A distributed force is any force where the **point of application** of the force is an **area** or a **volume**. This means that the "point of application" is not really a point at all. Though distributed forces are more difficult to analyze than point forces, distributed forces are quite common in real world systems so it is important to understand how to model them.

Distributed forces can be broken down into **surface forces** and **body forces**. Surface forces are distributed forces where the point of application is an area (a surface on the body). Body forces are forces where the point of application is a volume (the force is exerted on all molecules throughout the body). Below are some examples of surface and body forces.

Figure 4.4.1: The water pressure pushing on the surface of this dam is an example of a surface force. Image by Curimedia CC-BY-SA 2.0.

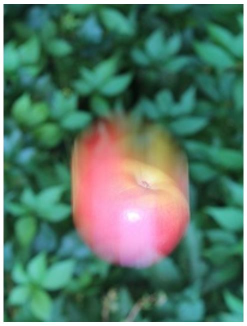


Figure 4.4.2: The gravitational force on this apple is distributed over the entire volume of the fruit. Gravitational forces are an example of body forces. Image by Zátonyi Sándor CC-BY 3.0.

Representing Distributed Forces:

Distributed forces are represented as a field of vectors. This is drawn as a number of discrete vectors along a line, over a surface, or over a volume, that are connected with a line or a surface as shown below.

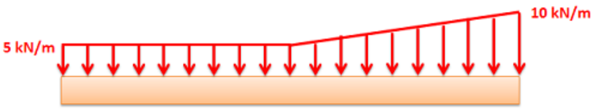


Figure 4.4.3: This is a representation of a surface force in a 2D problem (a force distributed over a line). The magnitude is given in units of force per unit distance.

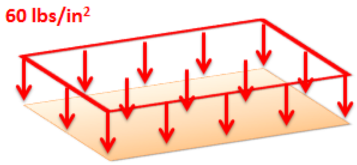


Figure 4.4.4: This is a representation of a surface force in a 3D problem (a force distributed over an area). The magnitude is given in units of force per unit area (also called a pressure).

Though these representations show a discrete number of individual vectors, there is actually a magnitude and direction at all points along the line, surface, or body. The individual vectors represent a sampling of these magnitudes and directions.

It is also important to realize that the magnitudes of distributed forces are given in force per unit distance, area, or volume. We must integrate the distributed force over its entire range to convert the force into the usual units of force.

Analyzing Distributed Forces:

For analysis purposes in statics and dynamics, we will usually substitute in a single point force that is statically equivalent to the distributed force in the problem. This single point force is called the **equivalent point load**, and it will cause the same accelerations or reaction forces as the distributed force while simplifying the math. However, in analysis that focuses on the strength of materials where the bodies are not rigid, this substitution will not work as the distributed forces will not cause the same deformations and stresses as the point force.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/VSIopUTg9kA.

4.5: Equivalent Point Load

An equivalent point load is a **single point force** that will have the **same effect** on a body as the original loading condition, which is usually a distributed force. The equivalent point load should always cause the same linear acceleration and angular acceleration as the original force it is equivalent to (or cause the same reaction forces if the body is constrained). Finding the equivalent point load for a distributed force often helps simplify the analysis of a system by removing the integrals from the equations of equilibrium or equations of motion in later analysis.

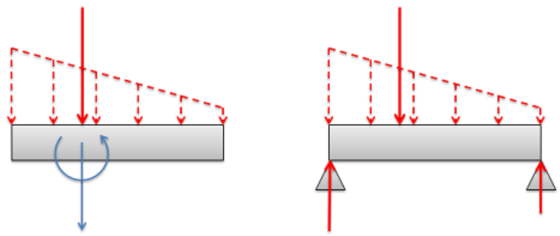


Figure 4.5.1: If the body is unconstrained as shown on the left, the equivalent point load (shown as a solid vector) will cause the same linear and angular acceleration as the original distributed load (shown with dashed vectors). If the body is constrained as on shown on the right, the equivalent point load (shown as a solid vector) will cause the same reaction forces as the original distributed force (shown with dashed vectors).

Finding the Equivalent Point Load

When finding the equivalent point load, we need to find the magnitude, direction, and point of application of a single force that is equivalent to the distributed force we are given. In this course we will only deal with distributed forces with a uniform direction, in which case the direction of the equivalent point load will match the uniform direction of the distributed force. This leaves the magnitude and the point of application to be found. There are two options available to find these values:

1. We can find the magnitude and the point of application of the equivalent point load **via integration** of the force functions.
2. We can use the **area/volume and the centroid/center of volume** of the area or volume under the force function.

The first method is more flexible, allowing us to find the equivalent point load for any force function that we can make a mathematical formula for (assuming we have the skill in calculus to integrate that function). The second method is usually faster, assuming that we can look up the values for the area or volume under the force curve and the values for the centroid or center of volume for the area under the curve.

Using Integration in 2D Surface Force Problems:

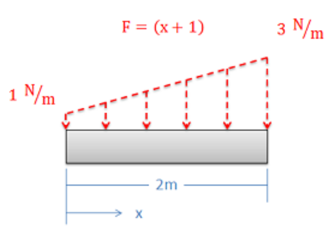


Figure 4.5.2: The block shown above has a distributed force acting on it. The force function relates the magnitude of the force to the x position along the top of the box.

Finding the equivalent point load via integration always begins by determining the mathematical formula that is the **force function**. The force function mathematically relates the magnitude of the force (F) to the position (x). In this case the force is acting along a single line, so the position can be entirely determined by knowing the x coordinate, but in later problems we may also need to relate the magnitude of the force to the y and z coordinates. In our example above, we can relate magnitude of the force to the position by stating that the magnitude of the force at any point in Newtons per meter is equal to the x position in meters plus one.

The magnitude of the equivalent point load will be equal to the area under the force function. This will be the integral of the force function over its entire length (in this case, from $x = 0$ to $x = 2$).

$$F_{eq} = \int_{x\,min}^{x\,max} F(x)\,dx \tag{4.5.1}$$

Now that we have the magnitude of the equivalent point load such that it matches the magnitude of the original force, we need to adjust the position (x_{eq}) such that it would cause the same **moment** as the original distributed force. The moment of the distributed force will be the integral of the force function $(F(x))$ times the moment arm about the origin (x). The moment of the equivalent point load will be equal to the magnitude of the equivalent point load that we just found times the moment arm for the equivalent point load (x_{eq}). If we set these two things equal to one another and then solve for the position of the equivalent point load (x_{eq}) we are left with the following equation.

$$x_{eq} = \frac{\int_{x\,min}^{x\,max} (F(x) * x)\,dx}{F_{eq}} \tag{4.5.2}$$

Now that we have the magnitude, direction, and position of the equivalent point load, we can draw the point load in our original diagram. This point force can be used in place of the distributed force in further analysis.

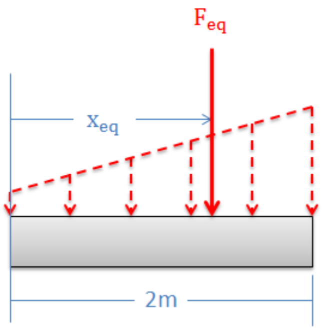


Figure 4.5.3: The values for F_{eq} and x_{eq} that we have solved for are the magnitude and position of the equivalent point load.

Using the Area and Centroid in 2D Surface Force Problems:

As an alternative to using integration, we can use the area under the force curve and the centroid of the area under the force curve to find the equivalent point load's magnitude and point of application respectively.

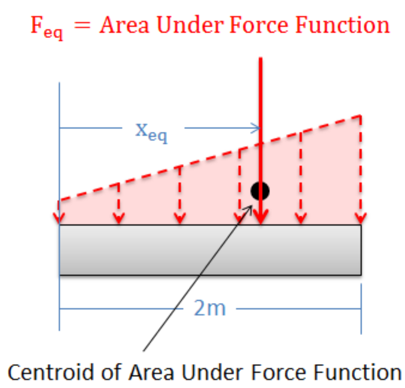


Figure 4.5.4: The magnitude of the equivalent point load is equal to the area under the force function. Also, the equivalent point load will travel through the centroid of the area under the force function.

The **magnitude** (F_{eq}) of the equivalent point load will be equal to the **area under the force function**. We can find this area using calculus, but there are often easier geometry-based ways of finding the area under the force function.

The equivalent point load will also **travel through centroid of the area under the force function**. This allows us to find the value for x_{eq}. The centroid for many common shapes can be looked up in tables, and the parallel axis theorem can be used to determine the centroid of more complex shapes (see the Appendix page on centroids for more details).

Using Integration in 3D Surface Force Problems:

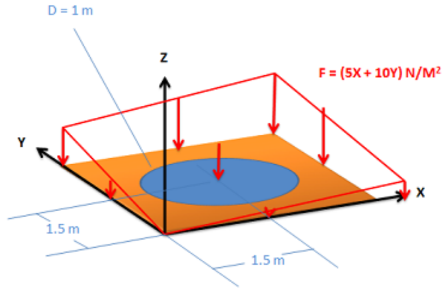


Figure 4.5.5: The magnitude of the surface force in this example varies with both x and y.

With surface force in a 3D problem, the force is distributed over a surface, rather than along a single line. To find the magnitude of the equivalent point load we will again start by finding the mathematical equation for the force function. Because the force is distributed over an area rather than just a line, the magnitude of the force may be related to both the x and the y coordinate, rather than just the x coordinate as before.

The magnitude of the equivalent point load (F_{eq}) will be equal to the volume under the force curve. To calculate this value we will integrate the force function over the area that the force is applied to. To integrate this function $F(x, y)$ in terms of the area, we will need to break the integral down further, integrating over x and then integrating over y.

$$F_{eq} = \int F(x, y)\, dA = \int_{y\ min}^{y\ max} \left(\int_{x\ min}^{x\ max} F(x, y)\, dx \right) dy \tag{4.5.3}$$

Once we solve for the magnitude of the equivalent point load, we can then solve for the position of the equivalent point load. Since the force is spread over a surface, we will need to calculate both the x (x_{eq}) and the y (y_{eq}) coordinates for the position. The process for solving for these values is similar to what was done with only an x value, except we change the moment arm value to match the equivalent point load coordinate we are looking for.

$$x_{eq} = \frac{\int (F(x, y) * x)\, dA}{F_{eq}} \tag{4.5.4}$$

$$y_{eq} = \frac{\int (F(x, y) * x)\, dA}{F_{eq}} \tag{4.5.5}$$

In each of the equations above, we will need to expand out the area integral into x and y integrals (as we did for F_{eq}) in order to be able to solve them.

Using Volume and Center of Volume in 3D Surface Force Problems:

Just as in the 2D problems, there are some available shortcuts to finding the equivalent point load in 3D surface force problems. For a force spread over an area, the **magnitude** (F_{eq}) of the equivalent point load will be equal to the **volume under the force function**. The equivalent point load will also **travel through the center of volume of the volume under the force function**. This should allow you to determine both x_{eq} and y_{eq}.

The center of volume for a shape will be the same as the center of mass for a shape if the shape is assumed to have uniform density. It should be possible to look these values up for common shapes in a table. Again, the parallel axis theorem can be used to find the center of volume for more complex shapes (See the Center of Mass page in Appendix 2 for more details).

Using Integration in Body Force Problems:

When we jump to body forces, the magnitude of our force will vary with x, y, and z coordinates. This means that our force function can include all of these variables $(F(x, y, z))$. To find the magnitude of the equivalent point load we integrate over the volume, breaking the volume integral down into x, y, and then z integrals.

$$F_{eq} = \int F(x,y,z)\,dV \tag{4.5.6}$$

$$= \int_{z\,min}^{z\,max} \left(\int_{y\,min}^{y\,max} \left(\int_{x\,min}^{x\,max} F(x,y,z)\,dx \right) dy \right) dz \tag{4.5.7}$$

To find the point of application of the equivalent point load, we will need to find all three coordinate positions. To do this, we will expand out the equations we used with two coordinates to include the third coordinate (z_{eq}).

$$x_{eq} = \frac{\int (F(x,y,z) * x)\,dV}{F_{eq}} \tag{4.5.8}$$

$$y_{eq} = \frac{\int (F(x,y,z) * y)\,dV}{F_{eq}} \tag{4.5.9}$$

$$z_{eq} = \frac{\int (F(x,y,z) * z)\,dV}{F_{eq}} \tag{4.5.10}$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/eikERyUNIOo.

Example 4.5.1

Determine the magnitude and the point of application for the equivalent point load of the distributed force shown below.

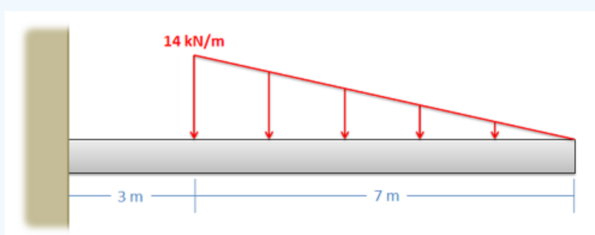


Figure 4.5.6: problem diagram for Example 4.5.1; a bar attached to a wall experiences a distributed force whose magnitude varies linearly over part of its length.

Solution

Video 4.5.2: Worked solution to example problem 4.5.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/h_E0XjIJaiI.

Example 4.5.2

Determine the magnitude and the point of application for the equivalent point load of the distributed force shown below.

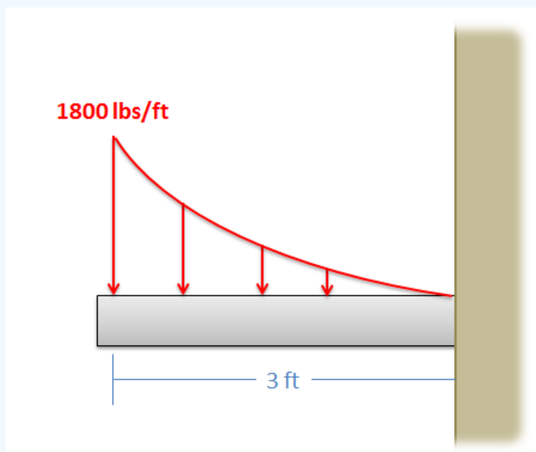


Figure 4.5.7: problem diagram for Example 4.5.2; a bar attached to a wall experiences a distributed force whose magnitude varies quadratically over its length.

Solution

Video 4.5.3: Worked solution to example problem 4.5.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/D4JoQpOyI38.

Example 4.5.3

Determine the magnitude and the point of application for the equivalent point load of the distributed force shown below.

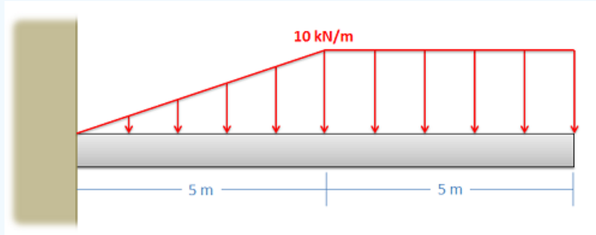


Figure 4.5.8: problem diagram for Example 4.5.3; a bar attached to a wall experiences a distributed force whose magnitude varies linearly over part of its length and remains constant for the remainder.

Solution

Video 4.5.4: Worked solution to example problem 4.5.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/UbnfNAQctyg.

Example 4.5.4

The wind has piled up sand into a corner on a building. The building supervisor is worried about the weight of the sand pushing against the roof of the basement below. The function describing force of the sand pushing down on the surface is given below. Find the magnitude, direction and point of application of the equivalent point load for the distributed force of the sand. Draw the equivalent point load in a diagram.

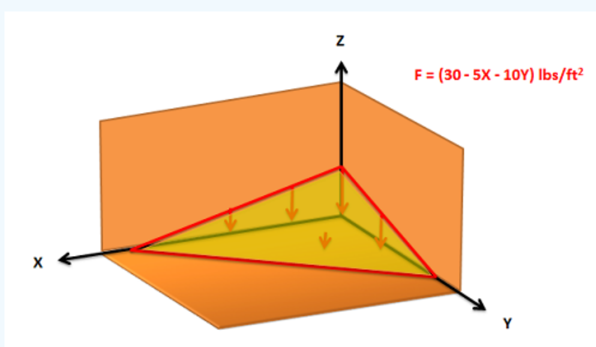


Figure 4.5.9: problem diagram for Example 4.5.4; sand accumulated in the corner of a building is assigned to a three-dimensional coordinate system and the distributed force it exerts on the floor beneath it is described with a force equation.

Solution

Video 4.5.5: Worked solution to example problem 4.5.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/IaSe_g3_Mgk.

4.6: Chapter 4 Homework Problems

Exercise 4.6.1

Determine if the two systems below are statically equivalent.

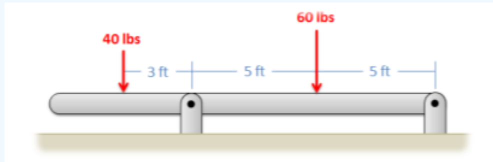


Figure 4.6.1: part 1 of the problem diagram for Exercise 4.6.1. A horizontal bar held off the ground by two supports experiences two point forces at different points along its length.

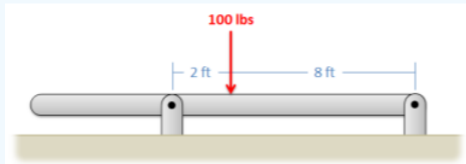


Figure 4.6.2: part 2 of the problem diagram for Exercise 4.6.1. A horizontal bar held off the ground by two supports experiences a single point force partway along its length.

Solution

No, they are not equivalent.

Exercise 4.6.2

Determine if the set of forces in A is statically equivalent to the set of forces and moments in B.

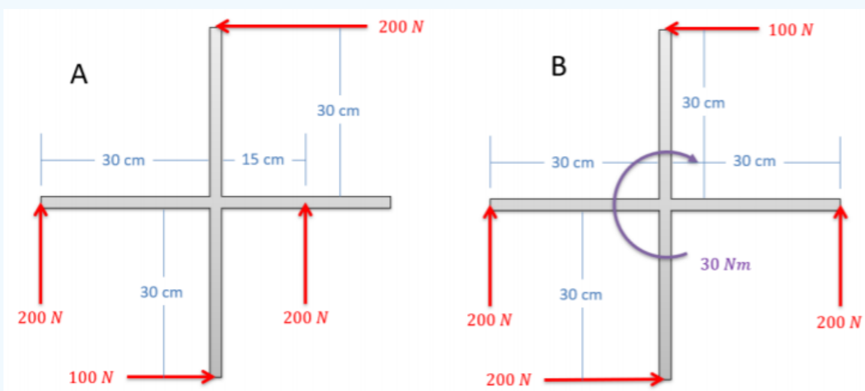


Figure 4.6.3: problem diagram for Exercise 4.6.2. Two uniform crosses of equal dimensions experience a set of forces (system A, left) and a set of forces and moments (system B, right).

Solution

No, they are not equivalent.

Exercise 4.6.3

Resolve the force shown below into a force and a couple acting at point A. Draw this force and couple on a diagram of the L-shaped beam.

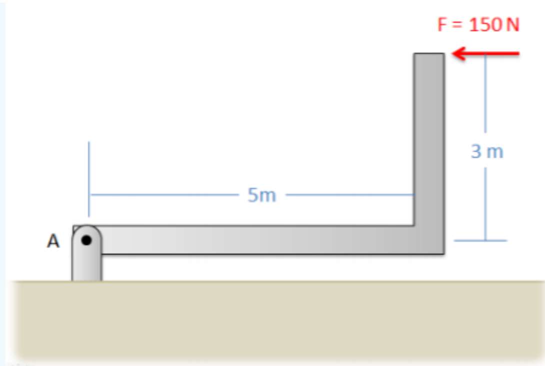


Figure 4.6.4: problem diagram for Exercise 4.6.3. An L-shaped beam is held parallel to the ground by a pin joint attached at one end (point A), with a force applied to the other free end of the L shape.

Solution

$F_A = 150\ N$ to the left

$M_A = 450\ Nm$

Exercise 4.6.4

Find the equivalent force couple system acting at point A for the setup shown below. Draw this force and couple on a diagram of the L-shaped beam.

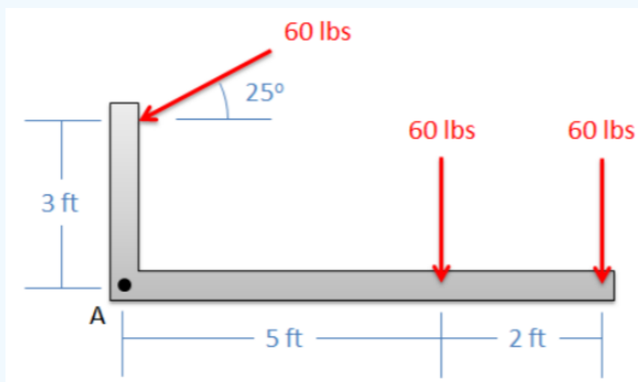


Figure 4.6.5: problem diagram for Exercise 4.6.4. An L-shaped beam, whose arms intersect at the point A, experiences several forces acting at different points along the arms.

Solution

$F_A = 155.2\ lbs,\ 69.5^{\circ}$ below the negative x-axis

$M_A = -556.9\ ft\ lbs$

Exercise 4.6.5

A helicopter is hovering with the wind force, the force from the tail rotor, and the moment due to drag shown below. Determine the equivalent force couple system at point C. Draw the final force and moment on a new diagram of the helicopter.

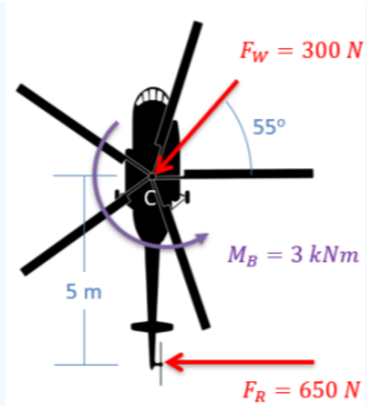


Figure 4.6.6: problem diagram for Exercise 4.6.5. A hovering helicopter with point C at the central hub of its main rotor experiences a force applied at C, a moment about C, and a force applied at a distance from C.

Solution

$F_{eq} = 858.01\ N$ acting at 16.6° below the negative x-axis

$M_{eq} = -250\ N\ m$

Exercise 4.6.6

Determine the equivalent point load (magnitude and location) for the distributed force shown below, **using integration.**

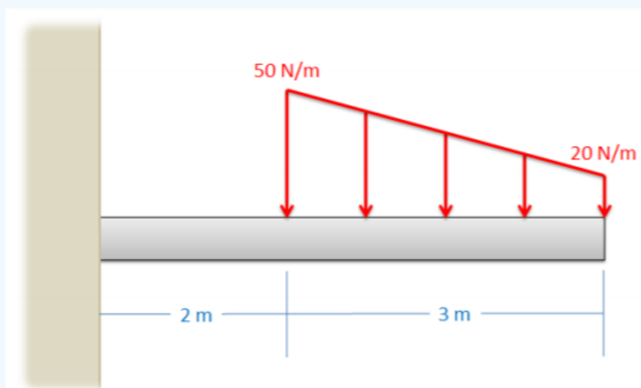


Figure 4.6.7: problem diagram for Exercise 4.6.6. A horizontal bar attached to a wall at one end experiences a distributed force, which varies linearly, over part of its length.

Solution

$F_{eq} = 105\ N$

$x_{eq} = 3.29\ m$ (measured from wall)

Exercise 4.6.7

Determine the equivalent point load (magnitude and location) for the distributed force shown below, **using integration.**

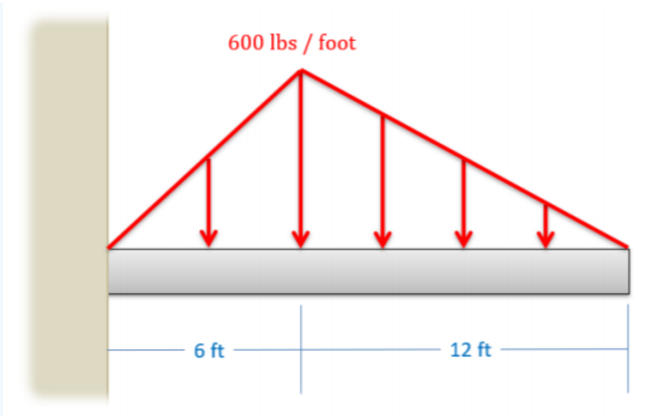


Figure 4.6.8: problem diagram for Exercise 4.6.7. A horizontal bar attached to a wall experiences a distributed force over its length, with magnitude varying linearly according to a piecewise force function.

Solution

$F_{eq} = 5400\, lbs$

$x_{eq} = 8\ ft$ (measured from wall)

Exercise 4.6.8

Use the **method of composite parts** to determine the magnitude and location of the equivalent point load for the distributed force shown below.

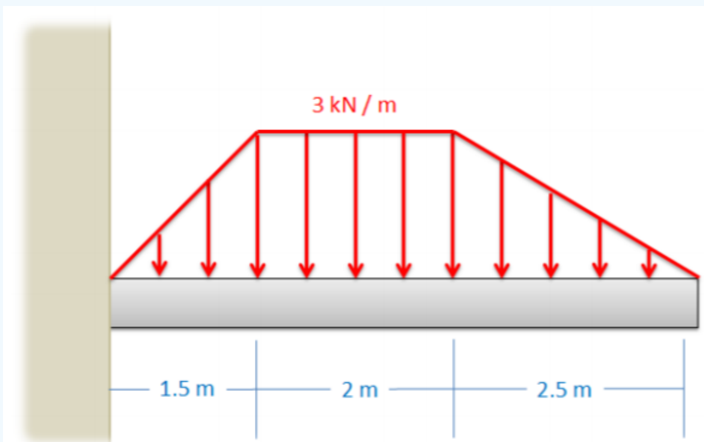


Figure 4.6.9: problem diagram for Exercise 4.6.8. A horizontal bar attached to a wall experiences a distributed force over its length, with magnitude varying linearly according to a piecewise force function.

Solution

$F_{eq} = 12\, kN$

$x_{eq} = 2.79\, m$ (measured from wall)

CHAPTER OVERVIEW

5: Engineering Structures

5.0: Video Introduction to Chapter 5

Video introduction to the topics to be covered in Chapter 5, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/BJ8SdEUnt2U.

5.1: Structures

An **engineering structure** is term used to describe any **set of interconnected bodies**. The different bodies in the structure can move relative to one another (such as the blades in a pair of scissors) or they can be be fixed relative to one another (such as the different beams connected to form a bridge).

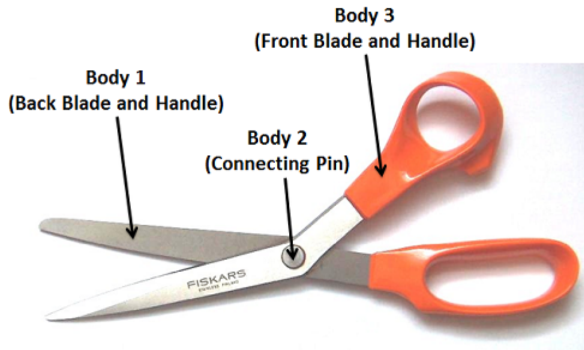


Figure 5.1.1: This pair of scissors is an example of an engineering structure. The structure consists of three different bodies that are interconnected. Adapted from Public Domain image by Comet27.

When analyzing engineering structures, we will sometimes analyze the structure as a whole, and we will sometimes break the structure down into individual bodies that are analyzed separately. The exact methods used depend upon what unknown forces we are looking for and what type of structure we are analyzing.

Internal and External Forces:

When examining a single body, we would find the forces that this body exerted on surrounding bodies, and the forces that these surrounding bodies would exert on the body we are analyzing. These forces are all considered **external forces** because they are forces between the body and the external environment.

In an engineering structure, we still have external forces where the structure is interacting with bodies external to the structure, but we also can think about the forces that different parts of the structure exert on one another (the force between the pin and the blade in Figure 5.1.1, for example). Since both of these bodies are part of the structure we are analyzing, these forces are considered **internal forces**.

If we only wish to determine the external forces acting on a structure, then we can treat the whole structure as a single body (assuming the structure is rigid as a whole). If we wish to determine the internal forces acting between components in the structure, then we will need to disassemble the structure into separate bodies in our analysis.

Types of Structures:

Another important consideration when analyzing structures is the type of structure that is being analyzed. All structures fall into one of three categories: **trusses**, **frames**, or **machines**. Frames and machines are analyzed in the same way so distinction between them is less important, but the analysis methods used for trusses vary greatly from the analysis methods used for frames and machines, so determining if a structure is a truss or not is an important first step in structure analysis.

Trusses:

A **truss** is a structure that consists **entirely** of **two-force members**. If any one body in the structure is not a two-force member, then the structure is either a frame or a machine. Also,in order to be a **statically determinate** truss (a truss where we can actually solve for all the unknowns), the truss must be independently rigid as a whole. If different parts of the truss could move relative to one another then the truss separated is not independently rigid.

Figure 5.1.2: This bridge is an example of a truss. It consists of a number of members connected at only two points (the ends of the beams). Public domain image by Leonard G.

A two-force member is a body where forces are applied at only two locations. If forces are applied at more than two locations, or if any moments are applied, then the body is not a two-force member (see the two-force member page for more details). Because of the unique assumptions we can make with two-force members, we can apply two unique methods to the analysis of trusses (the method of joints and the method of sections) that we cannot apply to frames and machines (where we cannot assume we have two-force members).

Frames and Machines:

A frame or a machine is structure where at least one component of the structure is not a two-force member. This component will be a body in the structure that has forces acting at three or more points on it. The difference between a frame and a machine is that a frame is rigid as a whole, while a machine is not rigid as a whole.

Figure 5.1.3: The legs of this stool have forces applied to them in three locations (the top, the cross beams, and the floor). The stool is also independently rigid, so this is a frame. Image by Besceh31 CC-BY-SA 2.5

Figure 5.1.4: Many pieces within this pair of locking pliers have forces applied to them at more than two locations. The pieces can also move relative to one another, so this is an example of a machine. Image by Duk CC-BY-SA 3.0

Because frames and machines do not consist entirely of two-force members, we cannot make the assumptions that allow us to use the method of joints and the method of sections. For this reason, we need to use a different analysis method (simply called the analysis of frames and machines here).

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/5Hz8h7qEVEM.

5.2: Two-Force Members

A **two-force member** is a body that has forces (and only forces, no moments) acting on it in only two locations. In order to have a two-force member in static equilibrium, the net force at each location must be equal, opposite, and collinear. This will result in all two-force members being in either tension or compression, as shown in the diagram below.

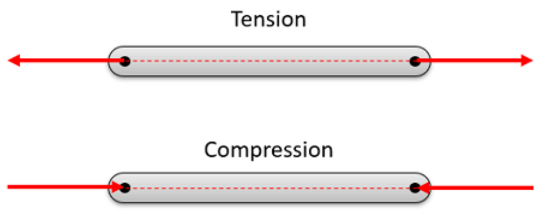


Figure 5.2.1: The forces acting on two-force members need to be equal, opposite, and collinear for the body to be in equilibrium.

Why the Forces Must Be Equal, Opposite and Collinear:

Imagine a beam where forces are only exerted at each end of the beam (a two-force member). The body has some non-zero force acting at one end of the beam, which we can draw as a force vector. If this body is in equilibrium, then we know two things:

1. the sum of the forces must be equal to zero, and
2. the sum of the moments must be equal to zero.

In order to have the sum of the forces equal to zero, the force vector on the other side of the beam must be equal in magnitude and opposite in direction. This is the only way to ensure that the sum of the forces is equal to zero with only two forces.

In order to have the sum of the moments equal to zero, the forces must be collinear. If the forces were not collinear, then the two equal and opposite forces would form a couple. This couple would exert a moment on the beam when there are no other moments to counteract the couple. Because the moment exerted by the two forces must be equal to zero, the perpendicular distance between the forces (d) must be equal to zero. The only way to achieve this is to have the forces be collinear.

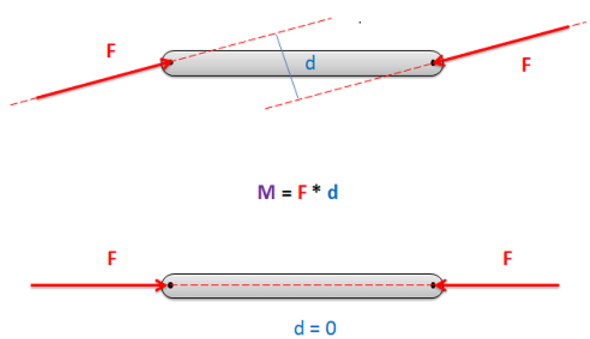


Figure 5.2.2: In order to have the sum of the moments be equal to zero, the forces acting on two-force members must always be collinear, acting along the line connecting the two points where forces are applied.

Why Two-Force Members Are Important:

By identifying two-force members, we greatly reduce the number of unknowns in our problem. In two-force members, we know that the forces must act along the line between the two connection points on the body. This means that the direction of the force vectors is known on either side of the body. Additionally, we know the forces are equal and opposite, so if we determine the magnitude and direction of the force acting on one side of the body, we automatically know the magnitude and direction of the force acting on the other side of the body.

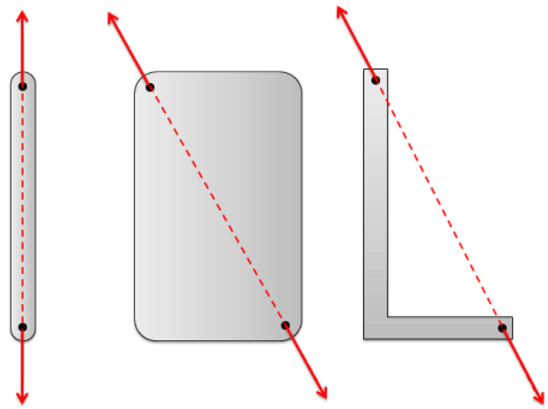


Figure 5.2.3: The forces in two-force members will always act along the line connecting the two points where forces are applied.

Two-force members are also important in distinguishing between trusses, and frames and machines. When we analyze trusses using either the method of joints or the method of sections, we will assume everything is a two-force member. If this assumption is incorrect, this will cause serious problems in the analysis. By making this assumption, though, we can use some shortcuts that will make truss analysis easier and faster than the analysis of frames and machines.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/m3GhMaZwHJw.

5.3: Trusses

A **truss** is an engineering structure that is made entirely of **two-force members**. In addition, statically determinate trusses (trusses that can be analyzed completely using the equilibrium equations) must be **independently rigid**. This means that if the truss was separated from its connection points, no one part would be able to move independently with respect to the rest of the truss.

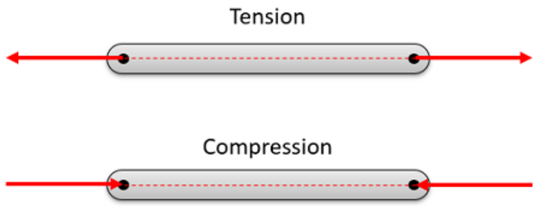


Figure 5.3.1: Trusses are made entirely of two-force members. This means that each member will either be in tension or compression, as shown here.

Trusses can be broken down further into **plane trusses** and **space trusses**. A plane truss is a truss where all members lie in a single plane. This means that plane trusses can essentially be treated as two-dimensional systems. Space trusses, on the other hand, have members that are not limited to a single plane. This means that space trusses need to be analyzed as a three-dimensional system.

Figure 5.3.2: The members of these trusses all lie in a single plane. These roof trusses are an example of a plane truss. Image by Riisipuuro CC-BY-SA 3.0.

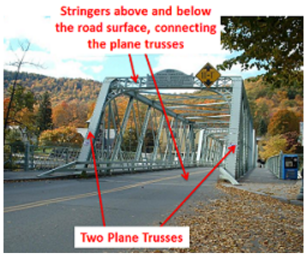


Figure 5.3.3: This bridge consists of two plane trusses connected by members called stringers. Adapted from image by ToddC4176 CC-BY-SA 3.0.

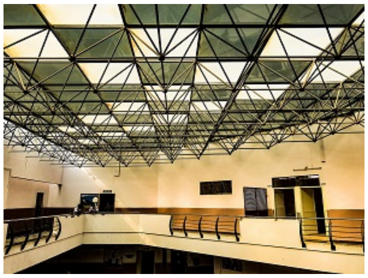


Figure 5.3.4: This roof supporting truss does not lie in a single plane. This is an example of a space truss. Image by IM3847 CC-BY-SA 4.0.

Figure 5.3.5: The power line tower also does not lie in a single plane and is therefore a space truss. Image by Anders Lagerås CC-BY-SA 2.5.

Analyzing Trusses:

When we talk about analyzing a truss, we are usually looking to identify not only the external forces acting on the truss structure, but also the forces acting on each member internally in the truss. Because each member of the truss is a two force member, we simply need to identify the magnitude of the force on each member, and determine if each member is in tension or compression.

To determine these unknowns, we have two methods available: the **method of joints**, and the **method of sections**. Both will give the same results, but each through a different process.

The method of joints focuses on the joints, or the connection points where the members come together. We assume we have a pin at each of these points that we model as a particle, we draw out the free body diagram for each pin, and then we write out the equilibrium equations for each pin. This will result in a large number of equilibrium equations that we can use to solve for a large number of unknown forces.

The method of sections involves pretending to split the truss into two or more different sections and then analyzing each section as a separate rigid body in equilibrium. In this method we determine the appropriate sections, draw free body diagrams for each section, and then write out the equilibrium equations for each section.

The method of joints is usually the easiest and fastest method for solving for all the unknown forces in a truss. The method of sections, on the other hand, is better suited to targeting and solving for the forces in just a few members without having to solve for all the unknowns. In addition, these methods can be combined if needed to best suit the goals of the problem solver.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/jn8sR-lS4tw.

5.4: Method of Joints

The **method of joints** is a process used to solve for the unknown forces acting on members of a **truss**. The method centers on the joints or connection points between the members, and it is usually the fastest and easiest way to solve for all the unknown forces in a truss structure.

Using the Method of Joints:

The process used in the method of joints is outlined below.

In the beginning, it is usually useful to label the members and the joints in your truss. This will help you keep everything organized and consistent in later analysis. In this book, the members will be labeled with letters and the joints will be labeled with numbers.

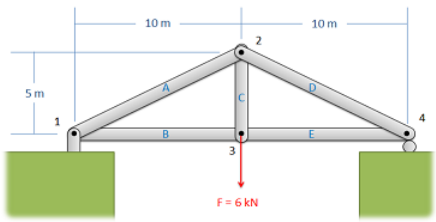


Figure 5.4.1: The first step in the method of joints is to label each joint and each member.

Treating the entire truss structure as a rigid body, draw a free body diagram, write out the equilibrium equations, and solve for the external reacting forces acting on the truss structure. This analysis should not differ from the analysis of a single rigid body.

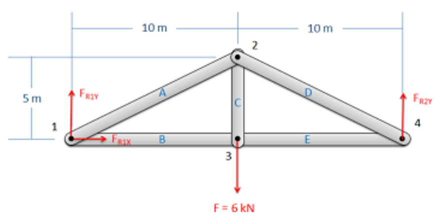


Figure 5.4.2: Treat the entire truss as a rigid body and solve for the reaction forces supporting the truss structure.

Assume there is a pin or some other small amount of material at each of the connection points between the members. Next you will draw a free body diagram for each connection point. Remember to include:

- Any external reaction or load forces that may be acting at that joint.
- A normal force for each two force member connected to that joint. Remember that for a two force member, the force will be acting along the line between the two connection points on the member. We will also need to guess if it will be a tensile or a compressive force. An incorrect guess now though will simply lead to a negative solution later on. A common strategy then is to assume all forces are tensile, then later in the solution any positive forces will be tensile forces and any negative forces will be compressive forces.
- Label each force in the diagram. Include any known magnitudes and directions and provide variable names for each unknown.

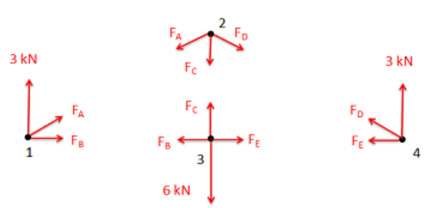


Figure 5.4.3: Drawing a free body diagram of each joint, we draw in the known forces as well as tensile forces from each two-force member.

- Write out the equilibrium equations for **each of the joints**. You should treat the joints as particles, so there will be force equations but no moment equations. With either two (for 2D problems) or three (for 3D problems) equations for each joint; this should give you a large number of equations.

- In planar trusses, the sum of the forces in the x direction will be zero and the sum of the forces in the y direction will be zero for each of the joints.

$$\sum \vec{F} = 0 \tag{5.4.1}$$

$$\sum F_x = 0\,; \;\sum F_y = 0 \tag{5.4.2}$$

- In space trusses, the sum of the forces in the x direction will be zero, the sum of the forces in the y direction will be zero, and the sum of forces in the z direction will be zero for each of the joints.

$$\sum \vec{F} = 0 \tag{5.4.3}$$

$$\sum F_x = 0\,; \;\sum F_y = 0\,; \;\sum F_z = 0 \tag{5.4.4}$$

- Finally, solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. If you assumed that all forces were tensile earlier, remember that negative answers indicate compressive forces in the members.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/B8SEG7xPI-o.

Example 5.4.1

Find the force acting in each of the members in the truss bridge shown below. Remember to specify if each member is in tension or compression.

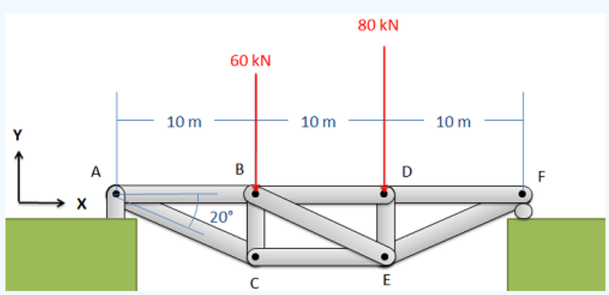


Figure 5.4.4: problem diagram for Example 5.4.1. A truss bridge represented as a 2D plane truss, with a standard-orientation xy-coordinate system.

Solution

Video 5.4.2: Worked solution to example problem 5.4.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/vowewkEdTzw.

Example 5.4.2

Find the force acting in each of the members of the truss shown below. Remember to specify if each member is in tension or compression.

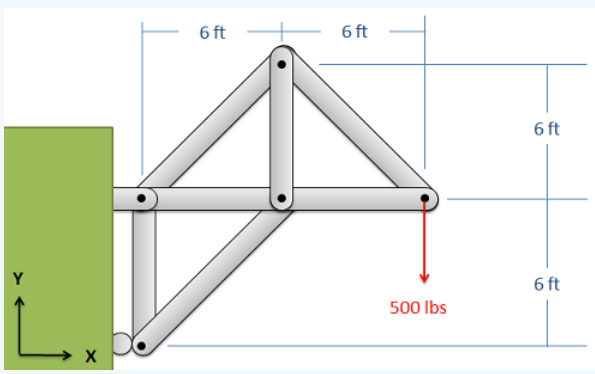


Figure 5.4.5: problem diagram for Example 5.4.2. A plane truss mounted on a wall, with a standard-orientation xy-coordinate system.

Solution

Video 5.4.3: Worked solution to example problem 5.4.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/IxnClZ-ppjM.

Example 5.4.3

Find the force acting in each of the members of the truss shown below. Remember to specify if each member is in tension or compression.

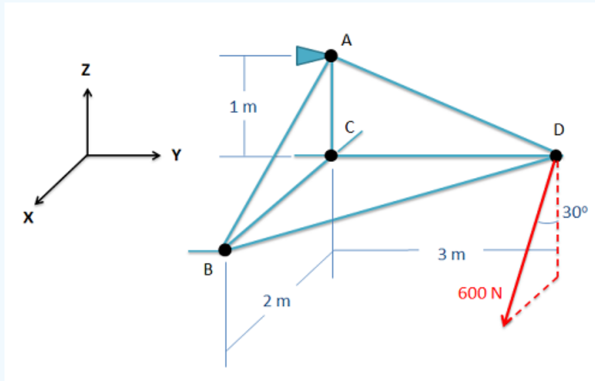


Figure 5.4.6: problem diagram for Example 5.4.3. A space truss supported by a single ball-and-socket joint, oriented on a 3D coordinate system with the yz-plane in the plane of the screen and the x-axis pointing out of the screen.

Solution

Video 5.4.4: Worked solution to example problem 5.4.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/sDKESSbufEk.

5.5: Method of Sections

The **method of sections** is a process used to solve for the unknown forces acting on members of a **truss**. The method involves breaking the truss down into individual sections and analyzing each section as a separate rigid body. The method of sections is usually the fastest and easiest way to determine the unknown forces acting in a specific member of the truss.

Using the Method of Sections:

The process used in the method of sections is outlined below.

1. In the beginning, it is usually useful to label the members in your truss. This will help you keep everything organized and consistent in later analysis. In this book, the members will be labeled with letters.

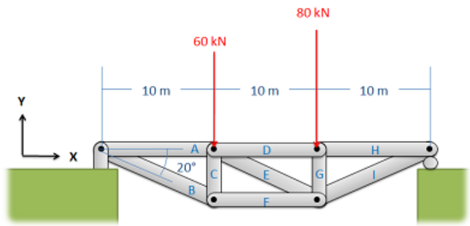


Figure 5.5.1: The first step in the method of sections is to label each member.

2. Treating the entire truss structure as a rigid body, draw a free body diagram, write out the equilibrium equations, and solve for the external reacting forces acting on the truss structure. This analysis should not differ from the analysis of a single rigid body.

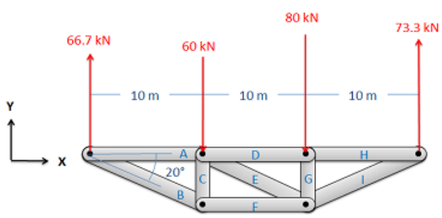


Figure 5.5.2: Treat the entire truss as a rigid body and solve for the reaction forces supporting the truss structure.

3. Next, you will imagine cutting your truss into two separate sections. The cut should travel through the member that you are trying to solve for the forces in, and should cut through as few members as possible. The cut does not need to be a straight line.

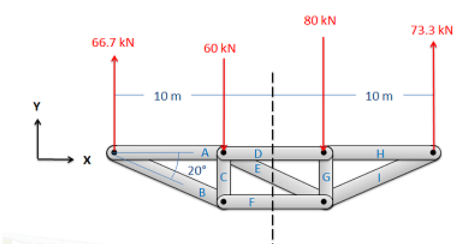


Figure 5.5.3: Next you will imagine cutting the truss into two parts. If you want to find the forces in a specific member, be sure to cut through that member. It also makes things easier if you cut through as few members as possible.

4. Next, you will draw a free body diagram for either one or both sections that you created. Be sure to include all the forces acting on each section.
   - Any external reaction or load forces that may be acting at the section.
   - An internal force in each member that was cut when splitting the truss into sections. Remember that for a two-force member, the force will be acting along the line between the two connection points on the member. We will also need to guess if it will be a tensile or a compressive force. An incorrect guess now, though, will simply lead to a negative solution later on. A common strategy then is to assume all forces are tensile; then later in the solution any positive forces will be tensile forces and any negative forces will be compressive forces.

- Label each force in the diagram. Include any known magnitudes and directions and provide variable names for each unknown.

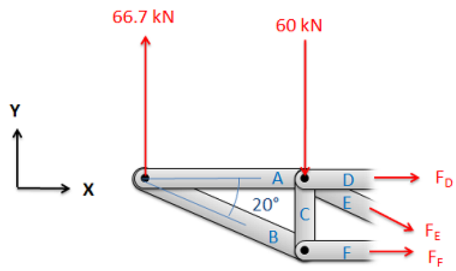


Figure 5.5.4: Next, draw a free body diagram of one or both halves of the truss. Add the known forces, as well as unknown tensile forces for each member that you cut.

5. Write out the equilibrium equations for each section you drew a free body diagram of. These will be extended bodies, so you will need to write out the force and the moment equations.
   - For 2D problems you will have three possible equations for each section: two force equations and one moment equation.

$$\sum \vec{F} = 0 \qquad \sum \vec{M} = 0 \tag{5.5.1}$$

$$\sum F_x = 0\,; \;\sum F_y = 0\,; \;\sum M_z = 0 \tag{5.5.2}$$

   - For 3D problems you will have six possible equations for each section: three force equations and three moment equations.

$$\sum \vec{F} = 0 \tag{5.5.3}$$

$$\sum F_x = 0\,; \;\sum F_y = 0\,; \;\sum F_z = 0 \tag{5.5.4}$$

$$\sum \vec{M} = 0 \tag{5.5.5}$$

$$\sum M_x = 0\,; \;\sum M_y = 0\,; \;\sum M_z = 0 \tag{5.5.6}$$

6. Finally, solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. If you assumed that all forces were tensile earlier, remember that negative answers indicate compressive forces in the members.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/XcRn776w22Q.

Example 5.5.1

Find the forces acting on members BD and CE. Be sure to indicate if the forces are tensile or compressive.

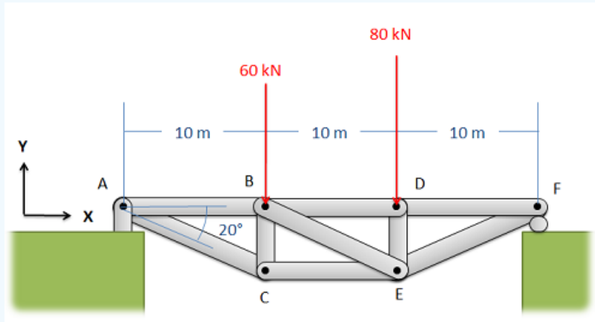


Figure 5.5.5: problem diagram for Example 5.5.1. A two-dimensional representation of a truss bridge, with a standard-orientation xy-coordinate system.

Solution

Video 5.5.2: Worked solution to example problem 5.5.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/9xxmHpLB1uU.

Example 5.5.2

Find the forces acting on members AC, BC, and BD of the truss. Be sure to indicate if the forces are tensile or compressive.

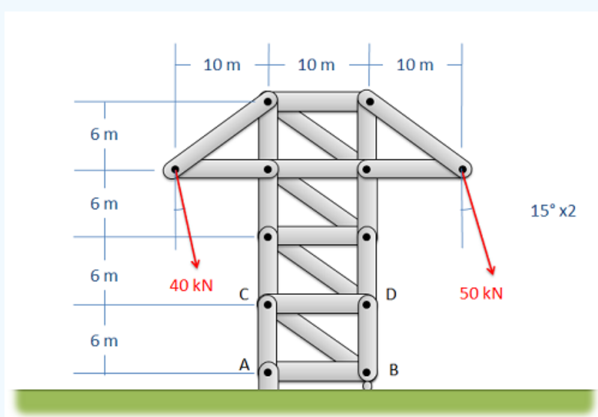


Figure 5.5.6: Problem diagram for Example 5.5.2. A two-dimensional representation of a tower composed of trusses, arranged in a tall rectangle made of rectangular subunits with a trapezoidal top.

Solution

Video 5.5.3: Worked solution to example problem 5.5.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Kp9U4d2qbvE.

5.6: Frames and Machines

A **frame** or a **machine** is an engineering structure that that contains at least one member that is not a two-force member.

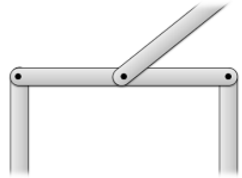


Figure 5.6.1: This horizontal beam is connected to other members (where normal forces would exist) at more than two locations. This beam is therefore not a two-force member.

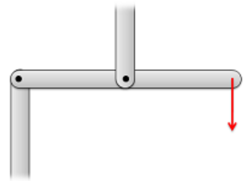


Figure 5.6.2: This beam has two connection points, but a force is acting on a third point. Therefore the beam has forces acting on it at more than two locations and it is not a two-force member.

A frame is a rigid structure, while a machine is not rigid. This means that no part can move relative to the other parts in a frame, while parts can move relative to one another in a machine. Though there is a difference in vocabulary in describing frames and machines, they are grouped together here because we use the same process to analyze both of these structures.

Figure 5.6.3: This stool contains non-two-force members (the legs) and no part can move relative to the other parts (it is rigid). Therefore it is a frame. Image by Besceh31 CC-BY-SA 2.5.

Figure 5.6.4: This pair of locking pliers contains non-two-force members and has parts that can move relative to one another (it is not rigid). Therefore it is a machine. Image by Duk CC-BY-SA 3.0.

Analyzing Frames and Machines:

When we talk about analyzing frames or machines, we are usually looking to identify both the external forces acting on the structure and the internal forces acting between members within the structure.

The method we use to analyze frames and machines (no special name here) centers around the process of breaking the structure down into individual components and analyzing each component as a rigid body. Where the components are connected, Newton's Third Law states that each body will exert an equal and opposite force on the other body. Each component will be analyzed as an independent rigid body leading to equilibrium equations for each component, but because of Newton's Third Law, some unknowns may show up acting on two bodies.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/QY-zWXzL7aI.

5.7: Analysis of Frames and Machines

The process used to analyze frames and machines involves breaking the structure down into individual components in order to solve for the forces acting on each component. **Sometimes** the structure as a whole can be analyzed as a rigid body, and each component can **always** be analyzed as a rigid body.

The Process for Analyzing Frames and Machines:

1. In the beginning it is usually useful to label the members in your structure. This will help you keep everything organized and consistent in later analysis. In this book, we will label everything by assigning letters to each of the joints.

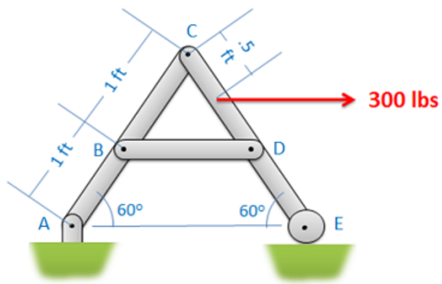


Figure 5.7.1: The first step in the analysis of frames and machines is to label the members.

2. Next you will need to determine if we can analyze the entire structure as a rigid body. In order to do this, the structure needs to be independently rigid. This means that it would be rigid even if we separated it from its supports. If the structure is independently rigid (no machines, and only some frames, will be independently rigid), then analyze the structure as a single rigid body to determine the reaction forces acting on the structure. If the structure is not independently rigid, skip this step.

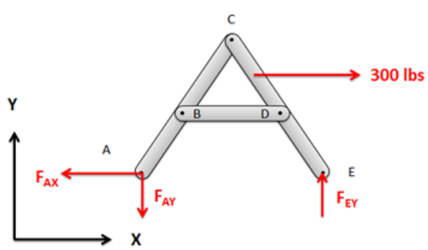


Figure 5.7.2: If, and only if, the structure is independently rigid, you should analyze the whole structure as a single rigid body to solve for the reaction forces.

3. Next you will draw a free body diagram for each of the components in the structure. You will need to include all forces acting on each member:
   - First, add any external reaction or load forces that may be acting on the components.
   - Second, identify any **two-force members** in the structure. At their connection points, they will cause a force with an unknown magnitude but a **known direction** (the forces will act along the line between the two connection points on the member).
   - Next, add in the reaction forces (and possibly moments) at the connection points between non-two-force members. For forces with an unknown magnitude and direction (such as in pin joints), the forces are often drawn in as having unknown x and y components (x, y and z for 3D truss problems).
   - Remember that the forces at each of the connection points will be a **Newton's Third Law pair**. This means that if one member exerts some force on some other member, then the second member will exert an equal and opposite force back on the first. When we draw out our unknown forces at the connection points, we must make sure that the forces acting on each member are **opposite in direction**.

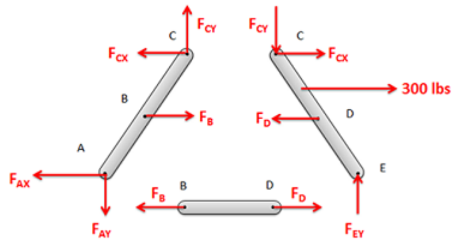


Figure \(\
PageIndex
{3}\):
Separate the structure into individual components and draw a free body diagram of each component. It is important to remember that the forces at each connection point are a Newton's Third Law pair.

4. Write out the equilibrium equations for each component you drew a free body diagram of. These will be extended bodies, so you will need to write out the force and the moment equations.
   - For 2D problems you will have three possible equations for each section: two force equations and one moment equation.

$$\sum \vec{F} = 0 \qquad \sum \vec{M} = 0 \tag{5.7.1}$$

$$\sum F_x = 0\,; \ \sum F_y = 0\,; \ \sum M_z = 0 \tag{5.7.2}$$

   - For 3D problems you will have six possible equations for each section: three force equations and three moment equations.

$$\sum \vec{F} = 0 \tag{5.7.3}$$

$$\sum F_x = 0\,; \ \sum F_y = 0\,; \ \sum F_z = 0 \tag{5.7.4}$$

$$\sum \vec{M} = 0 \tag{5.7.5}$$

$$\sum M_x = 0\,; \ \sum M_y = 0\,; \ \sum M_z = 0 \tag{5.7.6}$$

5. Finally, solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. If any force turns out to be negative, that indicates that the force actually travels in the opposite direction from what is indicated in your initial free body diagram.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/6DK6IOU_TmU.

Example 5.7.1

Find all the forces acting on each of the members in the structure below.

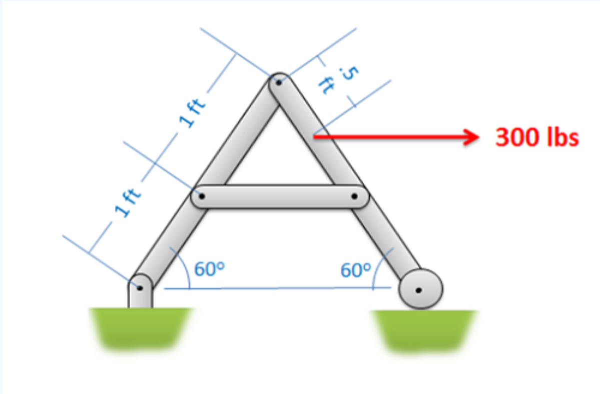


Figure 5.7.4: problem diagram for Example 5.7.1. A symmetrical A-shaped structure experiences an external force applied at one point.

Solution

Video 5.7.2: Worked solution to example problem 5.7.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/ix2BuRGMGBs.

Example 5.7.2

Find all the forces acting on each of the members in the structure below.

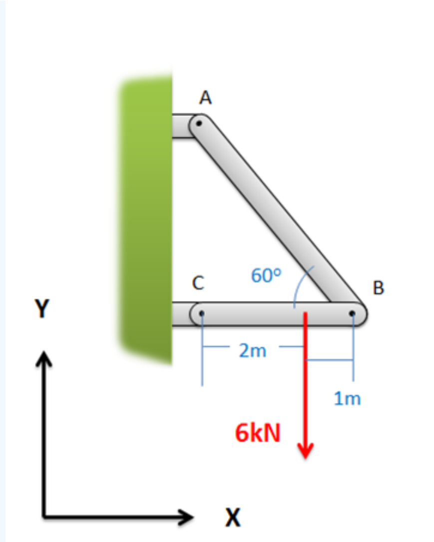


Figure 5.7.5: problem diagram for Example 5.7.2. A two-member structure is attached to a wall with pin joints and experiences an external force applied at one point.

Solution

Video 5.7.3: Worked solution to example problem 5.7.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/TJjsz5Yt3Y0.

Example 5.7.3

If two 150-Newton forces are exerted on the handles of the bolt cutter shown below, determine the reaction forces $(F_{R1}$ and (F_{R2}) exerted on the blades of the bolt cutter (this will be equal to the cutting forces exerted by the bolt cutters).

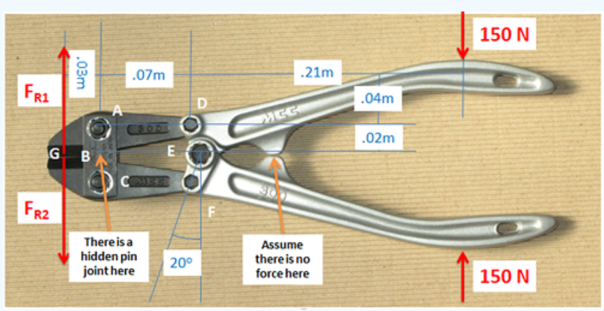


Figure 5.7.6: problem diagram for Example 5.7.3. Top-down view of a bolt cutter lying on a table facing left, divided into members by labeled points.

Solution

Video 5.7.4: Worked solution to example problem 5.7.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/uLsfSMzc5eQ.

Example 5.7.4

A 100-lb force is exerted on one side of a TV tray as shown below. Assuming there are no friction forces at the base, determine all forces acting on each of the three parts of the TV tray.

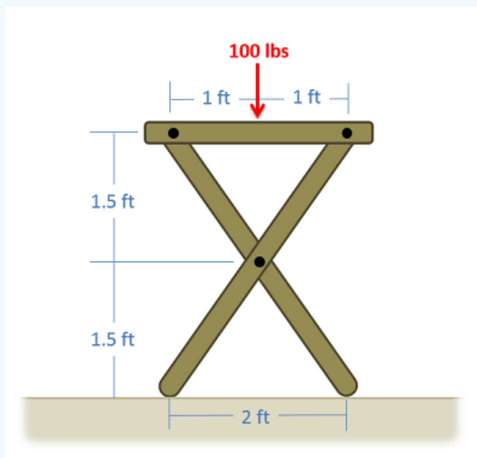


Figure 5.7.7: problem diagram for Example 5.7.4. Side view of a rectangular TV tray on two legs arranged in an X-shape, with a downwards force applied on the tray.

Solution

Video 5.7.5: Worked solution to example problem 5.7.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/qi7WNDSb43k.

5.8: Chapter 5 Homework Problems

Exercise 5.8.1

Use the method of joints to solve for the forces in each member of the lifting gantry truss shown below.

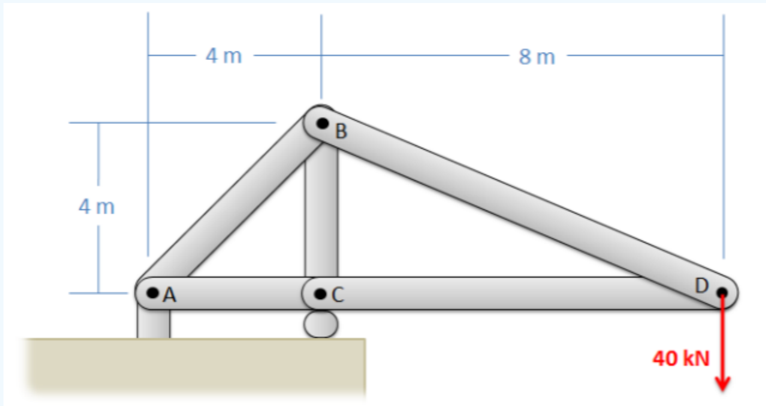


Figure 5.8.1: problem diagram for Exercise 5.8.1. A two-dimensional representation of a lifting gantry truss, which experiences a downwards force of 40 kN at one end.

Solution

$F_{AB} = 113.14\,\text{kN T}$, $F_{AC} = 80\ \text{kN C}$, $F_{BC} = 120\ \text{kN C}$

$F_{BD} = 89.44\,\text{kN T}$, $F_{CD} = 80\ \text{kN C}$

Exercise 5.8.2

The truss shown below is supported by two cables at A and E, and supports two lighting rigs at D and F, as shown by the loads. Use the method of joints to determine the forces in each of the members.

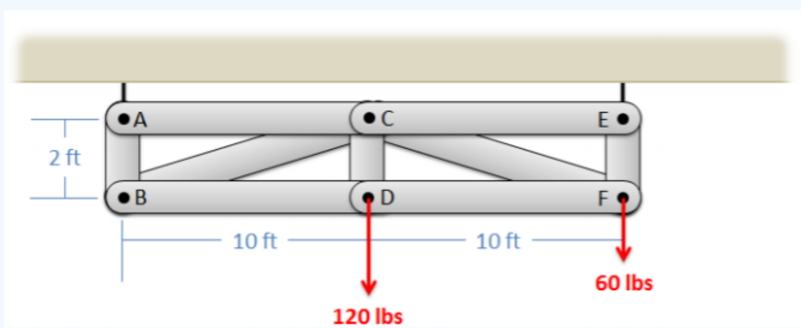


Figure 5.8.2: problem diagram for Exercise 5.8.2. A two-dimensional representation of a lighting support truss, which experiences the weights of lighting rigs hung at two points.

Solution

$F_{AB} = 60\ \text{lbs T}$, $F_{AC} = 0$, $F_{BC} = 305.94\,\text{lbs C}$

$F_{BD} = 300\ \text{lbs T}$, $F_{CD} = 120\ \text{lbs T}$, $F_{CE} = 0$

$F_{CF} = 305.94\,\text{lbs C}$, $F_{DF} = 300\ \text{lbs T}$, $F_{EF} = 120\ \text{lbs T}$

Exercise 5.8.3

The truss shown below is supported by a pin joint at A, a cable at D, and is supporting a 600 N load at point C. Use the method of joints to determine the forces in each of the members. Assume the mass of the beams are negligible.

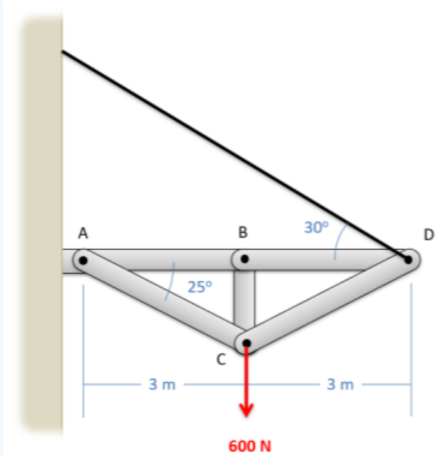


Figure 5.8.3: problem diagram for Exercise 5.8.3. A two-dimensional representation of a truss connected to the wall at one end, supported by a cable at the other, and experiencing a downwards force at one point.

Solution

$F_{AB} = 1162.97\text{N C}, F_{AC} = 709.86\,\text{N T}, F_{BC} = 0$

$F_{BD} = 1162.97\text{N C}, F_{CD} = 709.86\,\text{N T}$

Exercise 5.8.4

The space truss shown below is being used to lift a 250 lb box. The truss is anchored by a ball-and-socket joint at C (which can exert reaction forces in the x, y, and z directions) and supports at A and B that only exert reaction forces in the y direction. Use the method of joints to determine the forces acting all members of the truss.

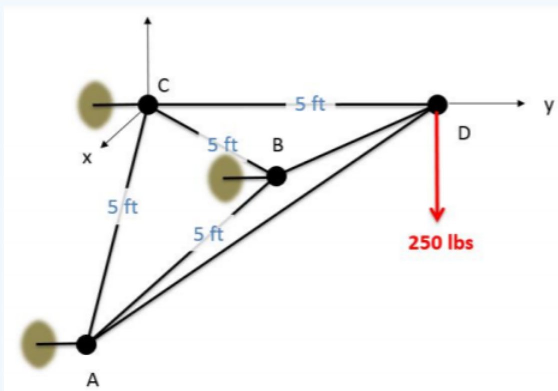


Figure 5.8.4: problem diagram for Exercise 5.8.4. A space truss attached to a wall at three points and supporting a load its free end.

Solution

$F_{AB} = 0, F_{AC} = 144.33\,\text{lbs T}, F_{AD} = 204.09\,\text{lbs C}$

$F_{BC} = 144.33\,\text{lbs T}, F_{BD} = 204.09\,\text{lbs C}, F_{CD} = 288.68\,\text{lbs T}$

Exercise 5.8.5

Use the method of sections to solve for the forces acting on members CE, CF, and DF of the gantry truss shown below.

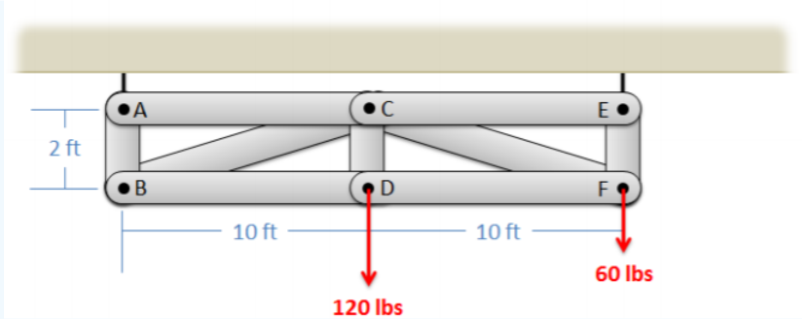


Figure 5.8.5: problem diagram for Exercise 5.8.5. A two-dimensional representation of a gantry truss, which experiences downwards forces applied at two points.

Solution

$F_{CE} = 0$, $F_{CF} = 306.2$ lbs C, $F_{DF} = 300.2$ lbs T

Exercise 5.8.6

You are asked to compare two crane truss designs as shown below. Find the forces in members AB, BC, and CD for Design 1 and find forces AB, AD, and CD for Design 2. What member is subjected to the highest loads in either case?

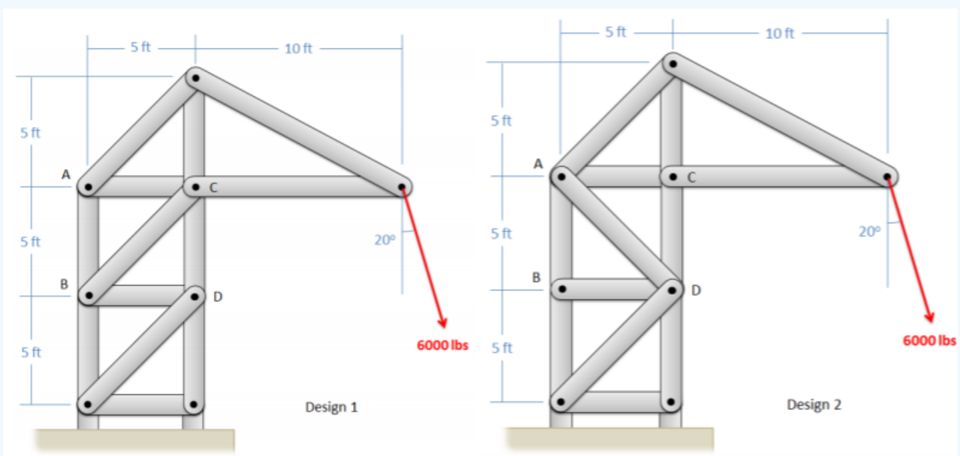


Figure 5.8.6: problem diagram for Exercise 5.8.6. Two versions of a crane truss design that differ only in the orientation of their support beams.

Solution

Design 1: $F_{AB} = 11,276$ lbs T, $F_{BC} = 2902$ lbs T, $F_{CD} = 18,967$ lbs C

Design 2: $F_{AB} = 13,322$ lbs T, $F_{AD} = 2902$ lbs C, $F_{CD} = 16,914$ lbs C

The largest forces are in member CD for both designs.

Exercise 5.8.7

The K truss shown below supports three loads. Assume only vertical reaction forces at the supports. Use the method of sections to determine the forces in members AB and FG. (Hint: you will need to cut through more than three members, but you can use your moment equations strategically to solve for exactly what you need).

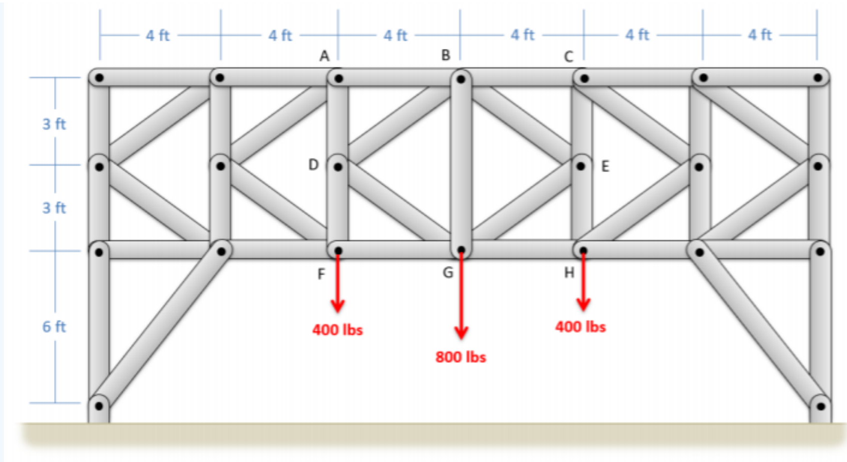


Figure 5.8.7: problem diagram for Exercise 5.8.7. A two-dimensional representation of a K-truss which experiences downwards forces applied at three points.

Solution

$F_{AB} = 1066.67\text{lbs C}$, $F_{FG} = 1066.67\text{lbs T}$

Exercise 5.8.8

The truss shown below is supported by a pin support at A and a roller support at B. Use the hybrid method of sections and joints to determine the forces in members CE, CF, and CD.

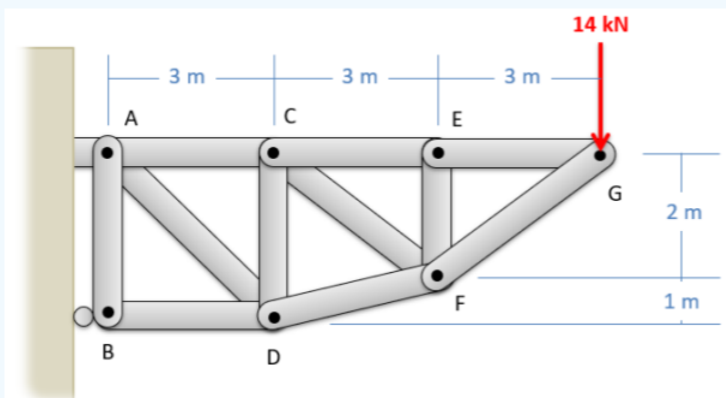


Figure 5.8.8: problem diagram for Exercise 5.8.9. A two-dimensional representation of a truss attached to a wall, experiencing a downwards force at one point.

Solution

$F_{CE} = 21\text{ kN T}$, $F_{CF} = 8.41\text{ kN T}$, $F_{CD} = 4.67\text{ kN C}$

Exercise 5.8.9

The shelf shown below is used to support a 50-lb weight. Determine the forces on members ACD and BC in the structure. Draw those forces on diagrams of each member.

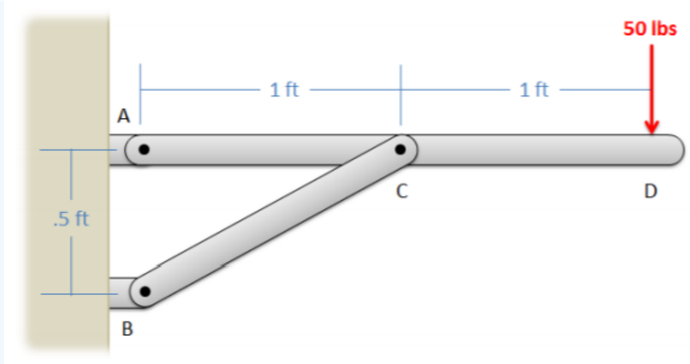


Figure 5.8.9: problem diagram for Example 5.8.9. Side view of a wall-mounted horizontal shelf supporting a weight at its free end.

Solution

$F_{BC} = 223.6\,\text{lbs}$ (Compression), $F_{A_X} = -200$ lbs, $F_{A_Y} = -50$ lbs

Exercise 5.8.10

A 20 N force is applied to a can-crushing mechanism as shown below. If the distance between points C and D is 0.1 meters, what are the forces being applied to the can at points B and D? (Hint: treat the can as a two-force member)

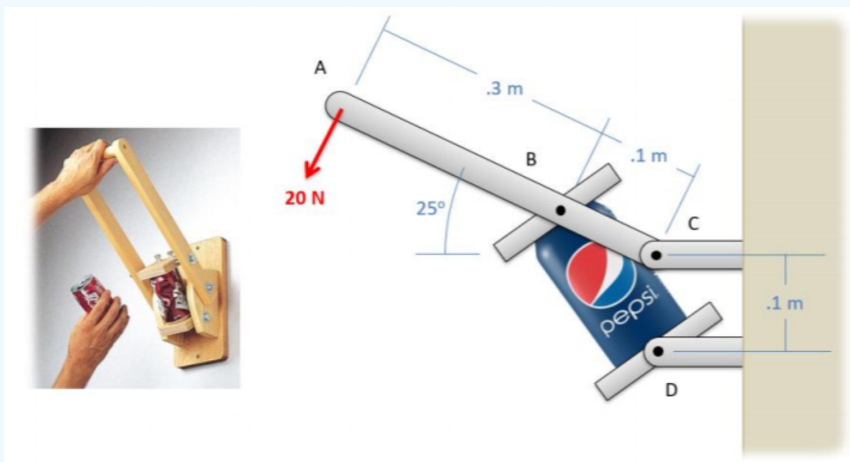


Figure 5.8.10: problem diagram for Exercise 5.8.10. Side view of a wall-mounted can-crushing mechanism that holds a soda can and experiences an applied force at its handle end.

Solution

$F_{can} = 148.9\,\text{N}$ (Compression)

Exercise 5.8.11

The suspension system on a car is shown below. Assuming the wheel is supporting a load of 3300 N and assuming the system is in equilibrium, what is the force we would expect in the shock absorber (member AE)? You can assume all connections are pin joints.

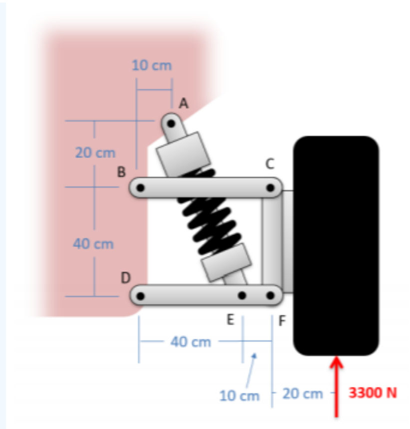


Figure 5.8.11: problem diagram for Exercise 5.8.11. Side view of the suspension system of one wheel of a car, experiencing an upwards force on the wheel.

Solution

$F_{AE} = 4611.9\,\text{N}$ (Compression)

Exercise 5.8.12

The chair shown below is subjected to forces at A and B by a person sitting in the chair. Assuming that normal forces exist at F and G, and that friction forces only act at point G (not at F), determine all the forces acting on each of the three members in the chair. Draw these forces acting on each part of the chair on a diagram.

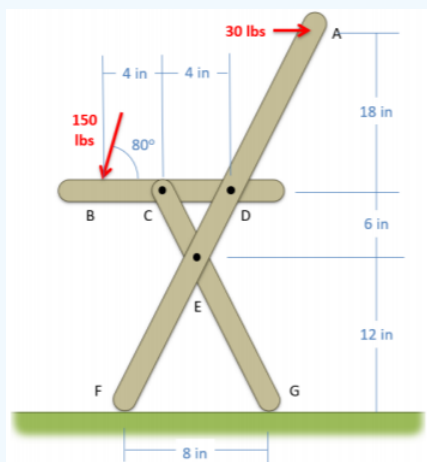


Figure 5.8.12: problem diagram for Exercise 5.8.12. Side view of a folding chair with a square seat, facing to the left, that experiences forces at two points from a person sitting in the chair.

Solution

$F_F = 108.3\,\text{lbs}$, $F_{G_X} = -3.95$ lbs, $F_{G_Y} = 39.5$ lbs

$F_{C_X} = \pm 16.89$ lbs, $F_{C_Y} = \pm 295.4$ lbs

$F_{D_X} = \pm 142.9$ lbs, $F_{D_Y} = \pm 147.7$ lbs

$F_{E_X} = \pm 112.9$ lbs, $F_{E_Y} = \pm 256.0$ lbs

CHAPTER OVERVIEW

6: Friction and Friction Applications

6.0: Video Introduction to Chapter 6

Video introduction to the topics covered in Chapter 6, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/ltBEDVYnFbE.

6.1: Dry Friction

Dry friction is the force that opposes one solid surface sliding across another solid surface. Dry friction always opposes the surfaces sliding relative to one another, and it can have the effect of either opposing motion or causing motion in bodies.

Figure 6.1.1: Dry friction occurs between the bottom of this training sled and the grassy field. The dry friction would oppose the motion of the sled along the field in this case. Image by Avenue CC-BY-SA 3.0

Figure 6.1.2: Dry friction occurs between the tires and the road for this motorcycle. The dry friction force for this motorcycle is what allows it to accelerate, decelerate, and turn. Public Domain image by Takisha Rappold.

The most commonly used model for dry friction is **coulomb friction**. This type of friction can further be broken down into static friction and kinetic friction. These two types of friction are illustrated in the diagram below. First, imagine a box sitting on a surface. A pushing force is applied parallel to the surface and is constantly being increased. A gravitational force, a normal force, and a frictional force are also acting on the box.

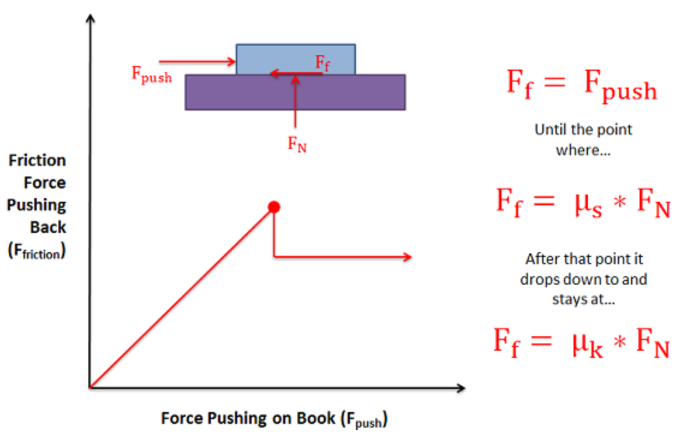


Figure 6.1.3: As the pushing force increases, the static friction force will be equal in magnitude and opposite in direction until the point of impending motion. Beyond this point, the box will begin to slip as the pushing force is greater in magnitude than the kinetic friction force opposing it.

Static friction occurs prior to the box slipping and moving. In this region, the friction force will be equal in magnitude and opposite in direction to the pushing force itself. As the magnitude of the pushing force increases, so does the magnitude of the friction force.

If the magnitude of the pushing force continues to rise, eventually the box will begin to slip. As the box begins to slip, the type of friction opposing the motion of the box changes from static friction to what is called kinetic friction. The point just before the box slips is known as **impending motion**. This can also be thought of as the maximum possible static friction force before slipping. The magnitude of the maximum static friction force is equal to the static coefficient of friction times the normal force existing between the box and the surface. This coefficient of friction is a property that depends on both materials and can usually be looked up in tables.

Kinetic friction occurs beyond the point of impending motion, when the box is sliding. With kinetic friction, the magnitude of the friction force opposing motion will be equal to the kinetic coefficient of friction times the normal force between the box and the surface. The kinetic coefficient of friction also depends upon the two materials in contact, but will almost always be less than the static coefficient of friction.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/OPRc38nDpKo.

Example 6.1.1

A 500 lb box is sitting on concrete floor. If the static coefficient of friction is 0.7 and the kinetic coefficient of friction is 0.6:

- What is the friction force if the pulling force is 150 lbs?
- What pulling force would be required to get the box moving?
- What is the minimum force required to keep the box moving once it has started moving?

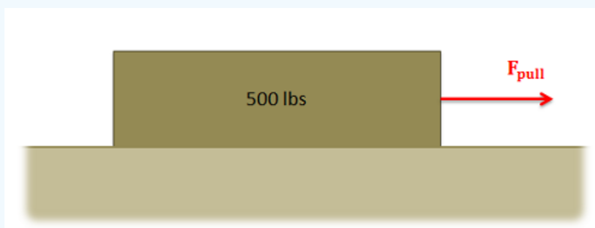


Figure 6.1.4: problem diagram for Example 6.1.1. A box experiences a pulling force towards the right.

Solution

Video 6.1.2: Worked solution to example problem 6.1.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/7R7kvKBxUjw.

Example 6.1.2

A 30 lb sled is being pulled up an icy incline of 25 degrees. If the static coefficient of friction between the ice and the sled is 0.4 and the kinetic coefficient of friction is 0.3, what is the required pulling force needed to keep the sled moving at a constant rate?

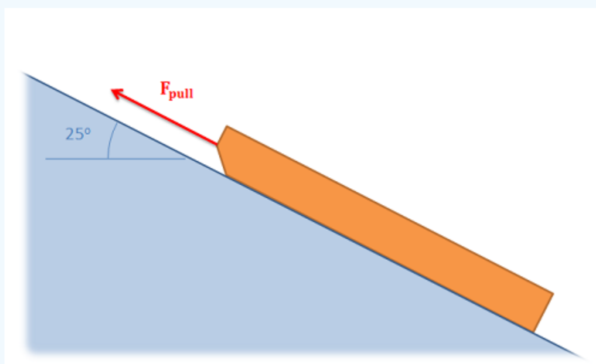


Figure 6.1.5: problem diagram for Example 6.1.2. A sled with a pulling force applied to move it up a 25° incline.

Solution

Video 6.1.3: Worked solution to example problem 6.1.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/oPMx-SZyiy4.

Example 6.1.3

A plastic box is sitting on a steel beam. One end of the steel beam is slowly raised, increasing the angle of the surface until the box begins to slip. If the box begins to slip when the beam is at an angle of 41 degrees, what is the static coefficient of friction between the steel beam and the plastic box?

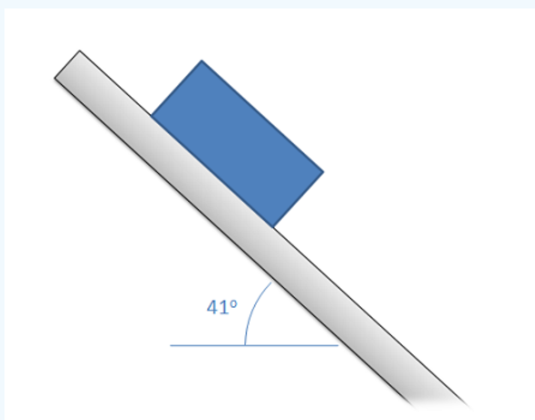


Figure 6.1.6: problem diagram for Example 6.1.3. A box on a steel beam whose left end has been raised until it is at 41° above the horizontal.

Solution

Video 6.1.4: Worked solution to example problem 6.1.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/ShGP5rzIHN4.

6.2: Slipping vs Tipping

Imagine a box sitting on a rough surface as shown in the figure below. Now imagine that we start pushing on the side of the box. Initially the friction force will resist the pushing force and the box will sit still. However, as we increase the force pushing the box one of two things will occur.

1. The pushing force will exceed the maximum static friction force and the box will begin to slide across the surface (slipping).
2. Or, the pushing force and the friction force will create a strong enough couple that the box will rotate and fall on its side (tipping).

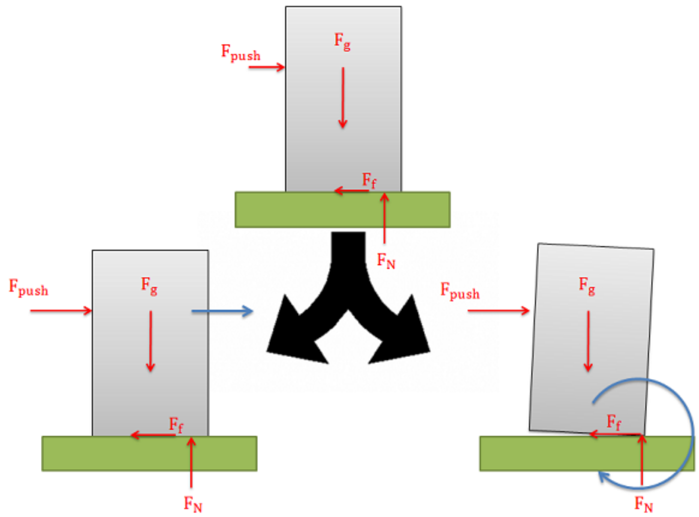


Figure 6.2.1: As the pushing force increases on the box, it will either begin to slide along surface (slipping) or it will begin to rotate (tipping).

When we look at cases where either slipping or tipping may occur, we are usually interested in finding which of the two options will occur first. To determine this, we usually determine both the pushing force necessary to make the body slide and the pushing force necessary to make the body tip over. **Whichever option requires less force is the option that will occur first.**

Determining the Force Required to Make an Object "Slip":

A body will slide across a surface if the pushing force exceeds the maximum static friction force that can exist between the two surfaces in contact. As in all dry friction problems, this limit to the friction force is equal to the static coefficient of friction times the normal force between the body. If the pushing force exceeds this value then the body will slip.

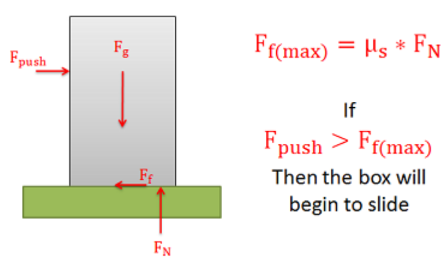


Figure 6.2.2: If the pushing force exceeds the maximum static friction force $(\mu_s * F_N)$ then the body will begin to slide.

Determining the Force Required to Make an Object "Tip":

The normal forces that support bodies are distributed forces. These forces will not only prevent the body from accelerating into the ground due to gravitational forces, but they can also redistribute themselves to prevent a body from rotating when forces cause a moment to act on the body. This redistribution will result in the equivalent point load for the normal force shifting to one side or the other. A body will tip over when the normal force can no longer redistribute itself any further to resist the moment exerted by other forces (such as the pushing force and the friction force).

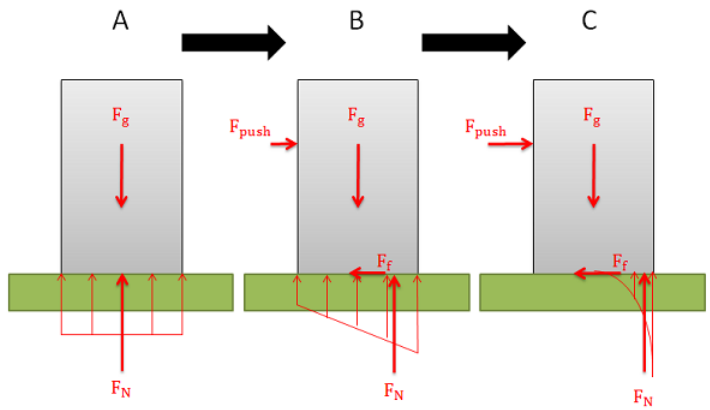


Figure 6.2.2: At rest (A) the normal force is a uniformly distributed force on the bottom of the body. As a pushing force is applied (B) the distributed normal force is redistributed, moving the equivalent point load to the right. This creates a couple between the gravity force and the normal force that will counter the couple exerted by the pushing force and the friction force. If the pushing force becomes large enough (C), the couple exerted by the gravitational force and the normal force will be unable to counter the couple exerted by the pushing force and the friction force.

The easiest way to think about the shifting normal force and tipping is to imagine the equivalent point load of the distributed normal force. As we push or pull on the body, the normal force will shift to the left or right. This normal force and the gravitational force create a couple that exerts a moment. This moment will be countering the moment exerted by the couple formed by the pushing force and the friction force.

Because the normal force is the direct result of physical contact, we cannot shift the normal force beyond the surfaces in contact (i.e., the edge of the box). If countering the moment exerted by the pushing force and the friction force requires shifting the normal force beyond the edge of the box, then the normal force and the gravity force will not be able to counter the moment and as a result the box will begin to rotate (i.e., tip over).

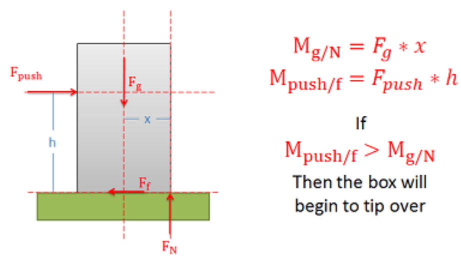


Figure 6.2.3: The body will tip when the moment exerted by the pushing and friction forces exceeds the moment exerted by the gravity and normal forces. For impending motion, the normal force will be acting at the very edge of the body.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/M2OZOkgVRBQ.

Example 6.2.1

The box shown below is pushed as shown. If we keep increasing the pushing force, will the box first begin to slide or will it tip over?

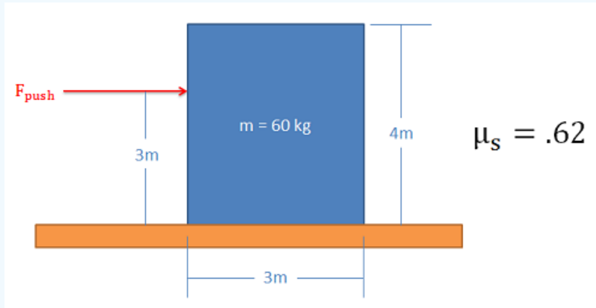


Figure 6.2.4: problem diagram for Example 6.2.1. A box on a flat surface, with a coefficient of static friction of 0.62 between the two surfaces, experiences a pushing force at a point on its left side.

Solution

Video 6.2.2: Worked solution to example problem 6.2.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/TeYgkfd4rTA.

Example 6.2.2

What is the maximum value of d that will allow the box to slide along the surface before tipping?

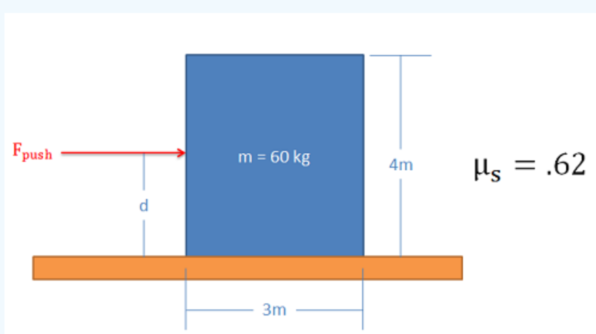


Figure 6.2.5: problem diagram for Example 6.2.2. A box on a flat surface, with a coefficient of static fricton of 0.62 between the two surfaces, experiences a pushing force at a point of unknown height (d on its left side.

Solution

Video 6.2.3: Worked solution to example problem 6.2.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/STDWwJsng6k.

6.3: Wedges

A **wedge** is a thin, inclined-plane-shaped object that is used to force two objects apart or to force one object away from a nearby surface. Wedges have the effect of allowing users to create very large normal forces to move objects with relatively small input forces. The friction forces in wedge systems also tend to be very large, though, and can reduce the effectiveness of wedges.

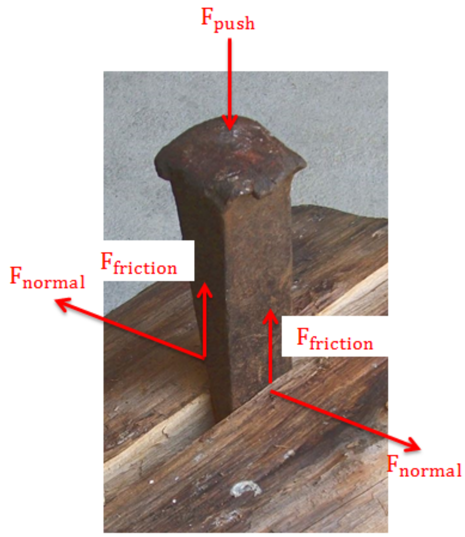


Figure 6.3.1: A hammer is used to push this wedge into the crack in this log. The normal forces are pushing the two halves of the log apart while the friction forces are opposing the pushing force. Adapted from image by Luigi Chiesa. CC-BY-SA-3.0.

To analyze a wedge system, we will need to draw free body diagrams of each of the bodies in the system (the wedge itself and any bodies the wedge will be moving). We need to be sure that we include the pushing force on the wedge, normal forces along any surfaces in contact, and friction forces along any surfaces in contact.

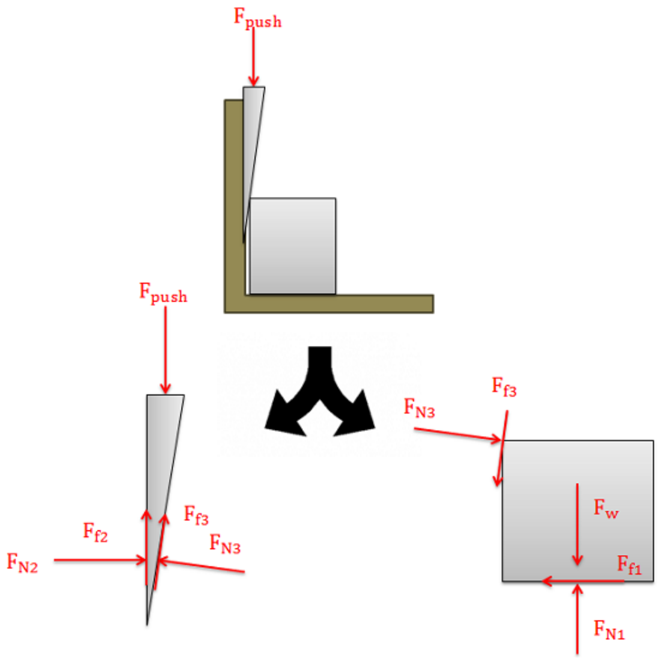


Figure 6.3.2: The top diagram shows a wedge being used to push a safe away from a wall. The first step in analyzing the system is to draw free body diagrams of the wedge and the safe. Remember that all normal forces will be perpendicular to the surfaces in contact and that all friction forces will be parallel to the surfaces in contact.

After we draw the free body diagram, we can work to simplify the problem. It is usually assumed that the wedge and the bodies will be sliding against one another, so each friction force will be equal to the kinetic coefficient of friction between the two surfaces times the associated normal force between the two forces. This reduces the number of unknowns and will usually allow us to solve for any unknown values.

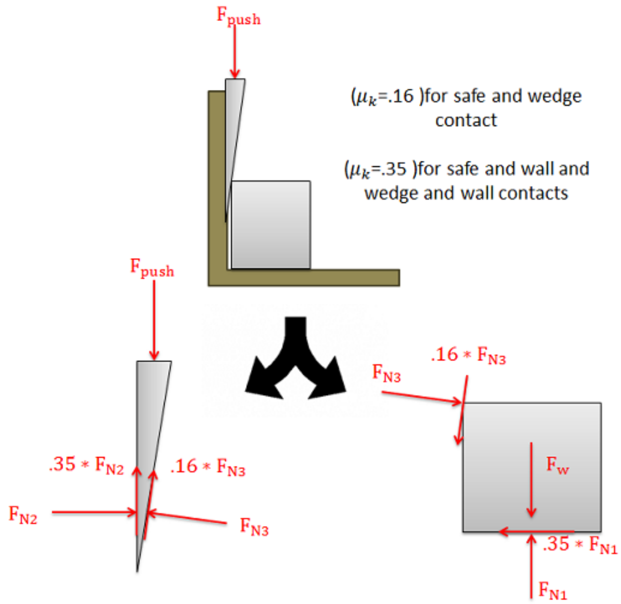


Figure 6.3.3: By replacing each of the friction forces with the kinetic coefficient of friction times the normal force, we can reduce the number of unknowns in our analysis.

With our simplified diagram, we will assume that the bodies are all in equilibrium and write out equilibrium equations for the two bodies. By solving the equilibrium equations, we can solve for any unknowns we have.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/5R3S5bYBULY.

Example 6.3.1

A heavy safe is being pushed away from a wall with a wedge as shown below. Assume the wedge has an angle of 5 degrees, the coefficient of friction (static and kinetic) between the wedge and the safe is 0.16, and the coefficients of friction (static and kinetic) between the wedge and the wall and the safe and the floor are both 0.35. What is the pushing force required to move the safe out from the wall?

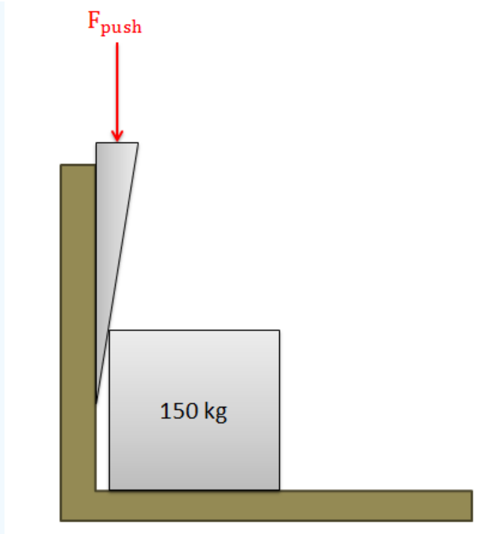


Figure 6.3.4: problem diagram for Example 6.3.1. A point-down wedge, with a pushing force applied at its base, is used to pry a safe away from a wall.

Solution

Video 6.3.2: Worked solution to example problem 6.3.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/r7Ut0GI8q00.

Example 6.3.2

A wedge as shown below is being used to lift the corner of the foundation of a house. How large must the pushing force be to exert a lifting force of one ton (2000 lbs)?

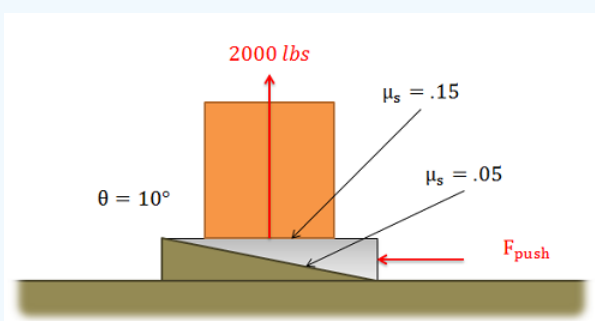


Figure 6.3.5: problem diagram for Example 6.3.2. A wedge is used to lift a block on a 10° incline with a force of 2000 lbs; coefficients of friction are given as 0.15 between the block and wedge, and 0.05 between the incline and the wedge.

Solution

Video 6.3.3: Worked solution to example problem 6.3.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/kgfh4Mna63M.

6.4: Power Screws

A power screw (also sometimes called a lead screw) is another simple machine that can be used to create very large forces. The screw can be thought of as a wedge or a ramp that has been wound around a shaft. By holding a nut stationary and rotating the shaft, we can have the nut sliding either up or down the wedge in the shaft. In this way, a relatively small moment on the shaft can cause very large forces on the nut.

Figure 6.4.1: The screw in this cider press is rotated with the handle on the top. The stationary nut on the frame of the press forces the shaft downward as it is turned. Public Domain image by Daderot.

Figure 6.4.2: By rotating the screw in this scissor jack, the user can move the nut (on the left end) closer or further from the anchor on the right end, which will raise or lower the car. Public Domain image by nrjfalcon1.

Static Analysis of Power Screws:

The easiest way to analyze a power screw system is to turn the problem into a 2D problem by "unwrapping" the ramp from around the shaft. To do this we will need two numbers. First we will need the **diameter** of the shaft, and second we will need either the **threads-per-inch/centimeter** or the **pitch** of the screw. The threads-per-inch tells you how many threads you have per inch/centimeter of screw. With a single thread design (most screws) this will also be the number of times the thread wraps around the screw in one inch/centimeter. The pitch, on the other hand, gives you the distance between two adjacent threads. Either of these numbers can be used to find the other.

Once we have these numbers, we can imagine unwrapping the ramp from around the screw and ending up with a ramp in one of the two situations below. In either case, we can use the inverse tangent function to find the **lead angle**, which can be thought of as the angle of the thread that the nut is climbing up. Finding the lead angle is the first step in analyzing a power screw system.

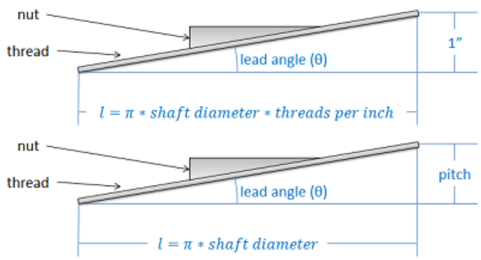


Figure 6.4.3: The lead angle of a screw is the angle of the thread that the nut will be climbing up as the shaft rotates.

Once we find the lead angle, we can draw a free body diagram of the "nut" in our unwrapped system. Here we include the pushing force which is pushing our nut up the incline, the load force which is the force the nut exerts on some external body, the normal force between the nut and screw, and the friction force between the nut and the screw.

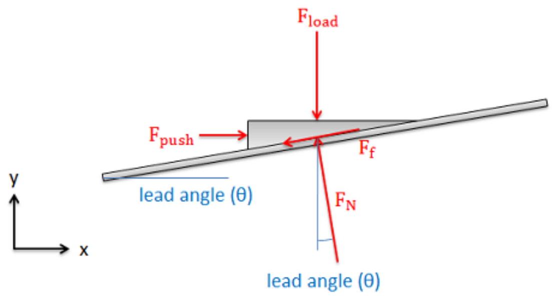


Figure 6.4.4: The free body diagram of the "nut" in our power screw system, with a standard-orientation xy-coordinate system.

If our screw is pushing a load at some constant rate, then we can assume two things: First, the nut is in equilibrium, so we can write out the equilibrium equations for the nut. Second, the nut is sliding, indicating that the friction force will be equal to the normal force times the kinetic coefficient of friction.

$$F_f = \mu_k * F_N \tag{6.4.1}$$

$$\sum F_x = F_{push} - F_N * \sin(\theta) - \mu_k * F_N * \cos(\theta) = 0 \tag{6.4.2}$$

$$\sum F_y = F_{push} - F_N * \cos(\theta) - \mu_k * F_N * \sin(\theta) = 0 \tag{6.4.3}$$

We can then simplify the equations above into a single equation relating the load force and the pushing force.

$$F_{push} = \frac{\sin(\theta) + \mu_k * \cos(\theta)}{\cos(\theta) - \mu_k * \sin(\theta)} * F_{load} \tag{6.4.4}$$

In reality, the pushing force is not a single force at all. It is the forces preventing the nut from rotating with the screw. The cumulative pushing force will really cause an equal and opposite moment to the input moment that is spinning the shaft.

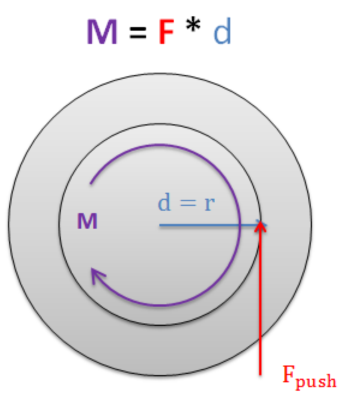


Figure 6.4.5: The pushing force is really just a representation of the forces keeping the nut from rotating. If the nut is not rotating, then these forces must cause an equal and opposite moment to the torque that is driving the screw.

Finally, if we replace the pushing force with the moment that is driving the screw in our system (in this case the torque T), we can relate the input torque that is driving our screw to the force that the nut on the screw is pressing forward with. Screw systems are usually designed to allow fairly small input moments to push very large load forces.

$$T = \frac{\sin(\theta) + \mu_k \cos(\theta)}{\cos(\theta) - \mu_k \sin(\theta)} * F_{load} * r_{shaft} \tag{6.4.5}$$

Self-Locking Screws

Imagine that we apply a torque to a power screw to lift a body; then when we get the load to the desired height we stop applying that torque to let the body sit where it is. If we were to redraw our free body diagram from earlier for the new situation, we would find two things.

1. The pushing force is missing (since a torque is no longer applied to the shaft).
2. The friction is now fighting against the nut sliding back down the ramp.

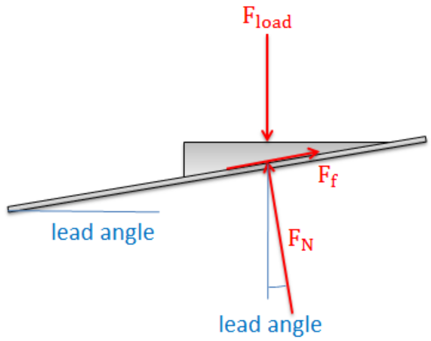


Figure 6.4.6: Without the pushing force from an applied torque, the friction force acts to prevent the nut from sliding down the ramp.

With this new free body diagram, there are two possible scenarios that could occur:

1. The friction force is large enough to keep the nut from sliding down the ramp, meaning everything will remain in static equilibrium if released.
2. The friction force will not be sufficient to keep the nut from sliding down the ramp, meaning that the load would begin to fall as soon as the torque is removed from the shaft.

With power screw applications such as a car jack, the second option could be very dangerous. It is therefore important to know if a power screw system is self-locking (scenario 1 above) or not self-locking (scenario 2 above).

To define the boundary between self-locking systems and non-self-locking systems, we use something called the **self-locking angle**. As intuition would tell us, slipping does not occur on very gentle slopes (small lead angles) while it does occur on very steep slopes (large lead angles). The angle at which the nut would begin to slip is known as the self-locking angle.

To find the self-locking angle, we will assume impending motion (relating the friction force to the normal force) and leave the lead angle as an unknown. This lets us create the free body diagram as shown below and gives us the equilibrium equations below.

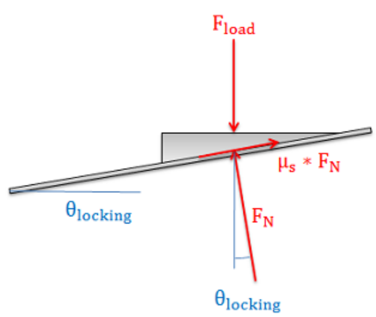


Figure 6.4.7: To find the self-locking angle we will assume impending motion. We then draw our free body diagram (above) and write out our equilibrium equations (below) accordingly.

$$\sum F_x = -F_N * \sin(\theta) + \mu_s * F_N * \cos(\theta) = 0 \tag{6.4.6}$$

$$\sum F_y = -F_{load} + F_N * \cos(\theta) + \mu_s * F_N * \sin(\theta) = 0 \tag{6.4.7}$$

Using the x equilibrium equation as a starting point, we can solve for the angle θ (eliminating the normal force all together in the process). This new equation shown below gives us the self-locking angle.

$$\theta_{locking} = \tan^{-1}(\mu_s) \tag{6.4.8}$$

Systems with lead angles smaller than this will be self-locking, while systems with lead angles larger than this will not be self-locking.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/mci-kH14JGw.

Example 6.4.1

The power screw below is being used to lift a platform with a weight of 12 pounds. Based on the information below...

- What is the required torque on the shaft to lift the load?
- Would the load fall if the toque was removed from the shaft?

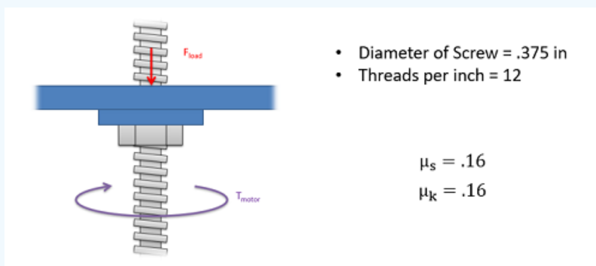


Figure 6.4.8: problem diagram for Example 6.4.1. A power screw with a 0.375-inch screw diameter, 12 threads per inch, is used to lift a platform, with a coefficient of static and kinetic friction of 0.16.

Solution

Video 6.4.2: Worked solution to example problem 6.4.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/uB2r3AtxCRs.

available upon request.

6.5: Bearing Friction

A bearing is the machine element used to support a rotating shaft. **Bearing friction** is the friction that exists between the rotating shaft and bearing that is supporting that shaft. Though many types of bearings exist (plain, ball, roller, hydrodynamic), in this course we will only be looking at plain bearings, also sometimes called journal bearings.

Figure 6.5.1: Some friction will exist in the bearings on this train car. Public Domain image by ds_30.

A plain bearing consists of a circular shaft fitted into a slightly larger circular hole as shown below. The shaft will usually be rotating and will be exerting some sort of load (F_{load}) onto the bearing. The bearing will then be supporting the shaft with some normal force, and a friction force will exist between the bearing surface and the surface of the shaft. Sometimes the rotation of the bearing will cause the shaft to climb up the side of the side of the bearing, causing the angle of the normal and friction forces to change, but this climbing is usually small enough that it is neglected.

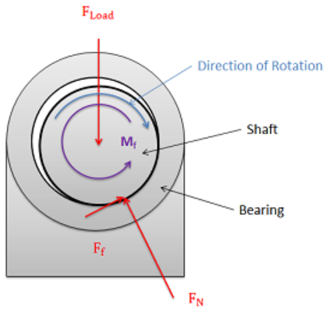


Figure 6.5.2: In a plain bearing like the one shown here, a friction force will oppose the rotation of the spinning shaft in the stationary bearing. This force will cause a small moment opposing the rotation of the shaft.

If we assume that the climbing in the bearing is negligible, the normal of the bearing on the shaft will be equal and opposite to loading force of the shaft on the bearing. Furthermore, if the shaft is rotating relative to the bearing, then the friction force will be equal to kinetic coefficient of friction times the normal force of the bearing on the shaft.

If the climb angle is assumed to be small, then...

$$F_{load} = F_N \tag{6.5.1}$$

$$F_f = \mu_k * F_N \tag{6.5.2}$$

The friction force will then be exerting a moment about the center of the shaft that opposes the rotation of the shaft. Unless some other moment is keeping the shaft spinning, this moment will eventually slow down and stop the rotation in the shaft. The magnitude of this moment will be equal to the magnitude of the friction force times the radius of the shaft.

$$M_f = r_{shaft} * (\mu_k * F_N) \tag{6.5.3}$$

Another important factor to keep in mind is that most bearings are lubricated. This can significantly lower the coefficients of friction (this is actually the main reason for using lubrication in machinery). When performing calculations, it is important to know if the bearing is lubricated and, if so, to use the appropriate friction coefficients.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/v8q6xnJN9zE.

Example 6.5.1

A cart and its load weigh a total of 200 lbs. There are two wheels on the cart, each with a diameter of 2 feet and each supporting half of the 200-lb load. The bearing attaching each wheel to the cart is a lubricated steel-on-steel (kinetic coefficient of friction = 0.05) journal bearing, with a shaft one inch in diameter. What is the moment due to friction from each bearing? Assuming there are no other sources of friction, what magnitude must the pulling force have in order to keep the cart moving at a constant speed?

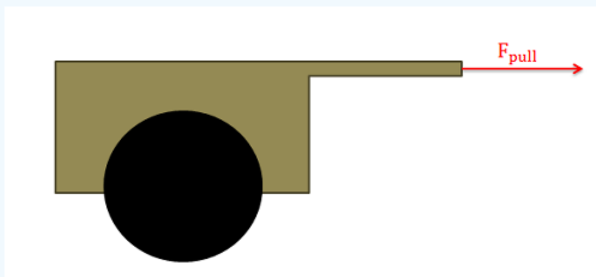


Figure 6.5.3: problem diagram for Example 6.5.1; a two-wheeled cart assembled with bearings is pulled by some force to the right.

Solution

Video 6.5.2: Worked solution to example problem 6.5.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/1R3BZIBQsXE.

available upon request.

6.6: Disc Friction

Disc friction is the friction that exists between a stationary surface and the end of a rotating shaft, or other rotating body. Disc friction will tend to exert a moment on the bodies involved, resisting the relative rotation of the bodies. Disc friction is applicable to a wide variety of designs including end bearings, collar bearings, disc brakes, and clutches.

Figure 6.6.1: This orbital sander rotates a circular sanding disc against a stationary surface. The disc friction between the sanding disc and the surface will exert a moment on both the surface and the sander. Image by Hedwig Storch CC-BY-SA 3.0.

Hollow Circular Contact Area

To start our analysis of disc friction we will use the example of a collar bearing. In this type of bearing, we have a rotating shaft traveling through a hole in a surface. The shaft is supporting some load force as shown, and a collar is used to support the shaft itself. In this case we will have a hollow circular contact area between the rotating collar and the stationary surface.

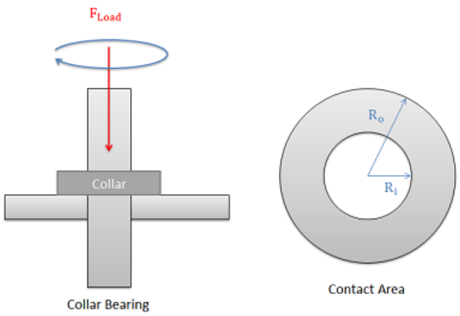


Figure 6.6.2: In a collar bearing we will have a hollow circular contact area between the rotating collar and the stationary surface.

The friction force at any point in the contact area will be equal to the normal force at that point times the kinetic coefficient of friction at that point. If we assume a uniform pressure between the collar and the surface and a uniform coefficient of friction, then we will have the same friction force exerted at all points. However, this does not translate into an equal moment exerted by each point. Points further from the center of rotation will exert a larger moment than points closer to the center of rotation because they will have a larger moment arm.

$$F_f = \mu_k F_N \tag{6.6.1}$$

$$F_i = F_o \tag{6.6.2}$$

$$M_i \neq M_o \tag{6.6.3}$$

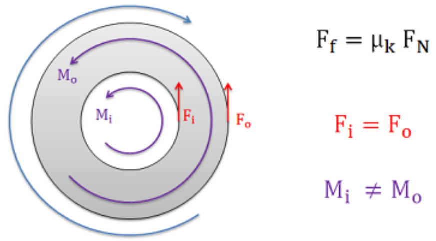


Figure 6.6.3: Though the friction forces at any point will be the same, points along the outside surface of the contact area will exert a larger moment than points along the inside surface of the contact area.

To determine the net moment exerted by the friction forces, we will need to use calculus to sum up the individual moments over the entire contact area. The moment at each individual point will be equal to the kinetic coefficient of friction, times the normal force pressure at that point (p), times the distance from that point to the center of rotation (r).

$$M = \int_A dM = \int_A (\mu_k)(p)(r)\, dA \tag{6.6.4}$$

To simplify things, we can move the constant coefficient of friction and the constant normal force pressure term outside the integral. We can also replace the pressure term with the load force on the bearing over the contact area. Finally, so that we can integrate over the range of R values, we can recognize that the rate of change in the area (dA) for the hollow circular areas is simply the rate of change of the r term (dr) times the circumference of the circle at r. These changes lead to the equation below.

$$M = \mu_k \left(\frac{F_{load}}{\pi(R_o^2 - R_i^2)} \right) \int_{R_i}^{R_o} r(2\pi r)\, dr \tag{6.6.5}$$

Finally, we can evaluate the integral from the inner radius to the outer radius. If we evaluate the integral and simplify, we will end up with the final equation below.

$$M = \frac{2}{3} \mu_k F_{load} \left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2} \right) \tag{6.6.6}$$

Solid Circular Contact Area

In cases where we have a solid circular contact area, such as a solid circular shaft in an end bearing or the orbital sander shown at the top of the page, we simply set the inner radius to zero and we can simplify the formula. If we do so, the original formula is reduced to the equation below.

$$M = \frac{2}{3} \mu_k F_{load} R_o \tag{6.6.7}$$

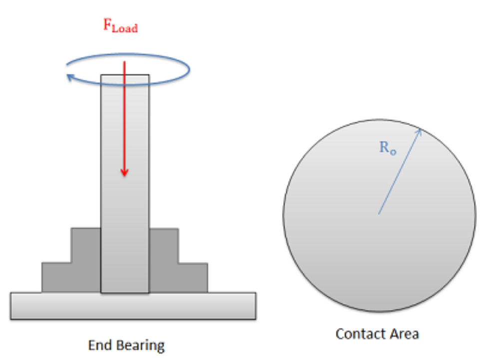


Figure 6.6.4: An end bearing has a solid circular contact area between the rotating shaft and the stationary bearing.

Circular Arcs (Disc Brakes)

In some cases, such as in disc brakes, we may have a contact area that looks like a section of the hollow circular contact area we had earlier. For this scenario we have less area on which the friction force can be exerted to cause a moment, but the smaller area also causes higher pressures in that contact area for the same load force. The end result is that these terms cancel each other out and we end up with the same formula we had for the hollow circular contact area when examining a single brake pad. Most disc brakes, however, have a pair of pads, one on each side of the rotating disc, so we will need to double the moment in our equation for the usual pair of pads.

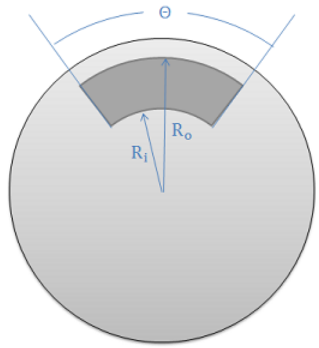


Figure 6.6.5: The contact area in disc brakes is often approximated as a circular arc with a given contact angle θ (theta).

$$\text{Single Brake Pad:} \quad M = \frac{2}{3}\mu_k\, F_{load}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right) \tag{6.6.8}$$

$$\text{Brake Pad on Each Side:} \quad M = \frac{4}{3}\mu_k\, F_{load}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right) \tag{6.6.9}$$

The calculations above show that the contact angle $theta$ is irrelevant to the stopping power of the brakes in theory. In practice, however, larger brake pads can slightly increase the stopping power of the brakes and provide other benefits such as better heat dissipation.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/jEtTe2cyumc.

Example 6.6.1

A disc sander is pressed against a wooden surface with a force of 50 N. Assuming the kinetic coefficient between the sanding pad and the wood is 0.6 and the diameter of the sanding disc is 0.2 meters, what is the torque the motor must exert to keep the disc spinning at a constant rate?

Figure 6.6.6: problem diagram for Example 6.6.1. Orbital sander being used to sand wood. Image by Hedwig Storch CC-BY-SA 3.0.

Solution

Video 6.6.2: Worked solution to example problem 6.6.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/0cBfIfDYDic.

Example 6.6.2

In the disc brake setup shown below, a pair of brake pads is pressed into the rotor with a force of 300 lbs. If the kinetic coefficient of friction between the brake pads and the rotor is 0.4, find the stopping torque exerted by the brake pads.

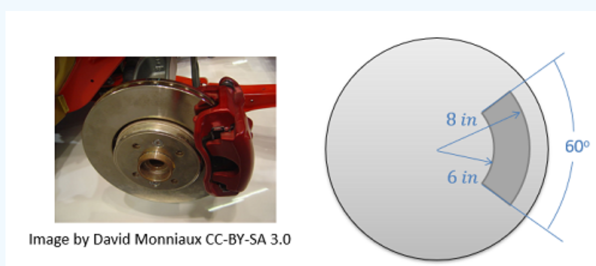


Figure 6.6.7: problem diagram for Example 6.6.2. Image of a disc brake and diagram of its contact area.

Solution

Video 6.6.3: Worked solution to example problem 6.6.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/lYrN0zr6kF8.

6.7: Belt Friction

In any system where a belt or a cable is wrapped around a pulley or some other cylindrical surface, we have the potential for friction between the belt or cable and the surface it is in contact with. In some cases, such as a rope over a tree branch being used to lift an object, the friction forces represent a loss. In other cases such as a belt-driven system, these friction forces are put to use transferring power from one pulley to another pulley.

Figure 6.7.1: In many belt-driven systems, the belt friction keeps the pulley from slipping relative to the belt. This allows us to use belts to transfer forces from one pulley to another pulley. Image by Kiilahihnakone, CC-BY-SA 3.0.

Figure 6.7.2: If we were to pass a rope over a tree branch to help lift an object such as this bear bag, the rope would experience belt friction resisting the sliding of the rope relative to the surface of the tree branch. Image by Virginia State Park Staff, CC-BY 2.0.

For analysis, we will start a flat, massless belt passing over a cylindrical surface. If we have an equal tension in each belt, the belt will experience a non-uniform normal force from the cylinder that is supporting it.

In a frictionless scenario, if we were to increase the tension on one side of the rope it would begin to slide across the cylinder. If friction exists between the rope and the surface though, the friction force will oppose with sliding motion, and prevent it up to a point.

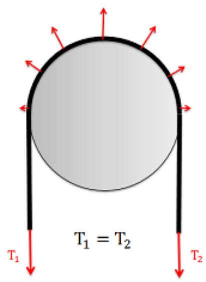


Figure 6.7.3: With equal tensions on each side of the belt, only a non-uniform normal force exists between the belt and the surface.

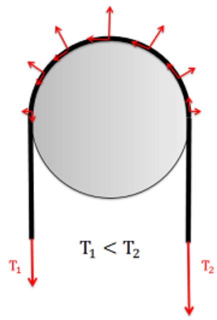


Figure 6.7.4: With unequal tensions, a friction force will also be present opposing the relative sliding of the belt to the surface.

Friction in Flat Belts

A flat belt is any system where the pulley or surface only interacts with the bottom surface of the belt or cable. If the belt or cable instead fits into a groove, then it is considered a V belt.

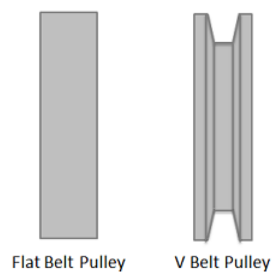


Figure 6.7.5: For a flat belt, the belt or cable will interact with the bottom surface. For a V belt, the belt or cable will interact with the sides of a groove.

When analyzing systems with belts, we are usually interested in the range of values for the tension forces where the belt will not slip relative to the surface. Starting with the smaller tension force on one side (T_1), we can increase the second tension force (T_2) to some maximum value before slipping. For a flat belt, the maximum value for T_2 will depend on the value of T_1, the static coefficient of friction between the belt and the surface, and the contact angle between the belt and the surface (β) given in radians, as described in the equation below.

$$T_{2_{max}} = T_1\, e^{\mu_s \beta} \tag{6.7.1}$$

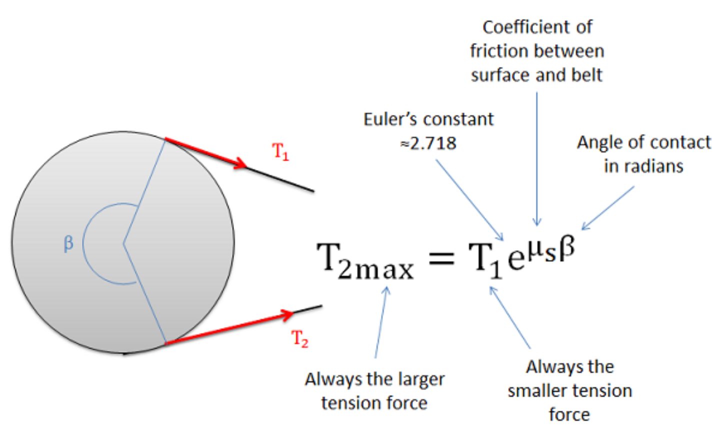


Figure 6.7.6: The method for determining the maximum value of T_2 before the belt starts slipping.

Friction in V Belts

A V belt is any belt that fits into a groove on a pulley or surface. For the V belt to be effective, the belt or cable will need to be in contact with the sides of the groove, but not the base of the groove as shown in the diagram below. With the normal forces on each side, the vertical components must add up the the same as what the flat belt would have, but the added horizontal components of the normal forces, which cancel each other out, increase the potential for friction forces.

The equation for the maximum difference in tensions in V belt systems is similar to the equation in flat belt systems, except we use an "enhanced" coefficient of friction that takes into account the increased normal and friction forces possible because of the groove.

$$T_{2_{max}} = T_1\, e^{\mu_{s\,(enh)}\beta}, \quad \text{where} \tag{6.7.2}$$

$$\mu_{s\,(enh)} = \frac{\mu_s}{\sin\left(\frac{\alpha}{2}\right)} \tag{6.7.3}$$

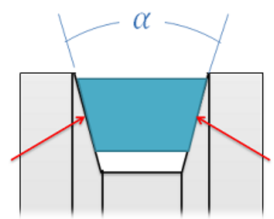


Figure 6.7.7: In a V belt, the "enhanced" coefficient of friction takes into account the coefficient of friction between the two materials as well as the groove angle.

As we can see from the equation above, steeper sides to the groove (which would result in a smaller angle α) result in an increased potential difference in the tension forces. The tradeoff with steeper sides, however, is that the belt becomes wedged in the groove and will require force to unwedge itself from the groove as it leaves the pulley. This will cause losses that decrease the efficiency of the belt driven system. If very high tension differences are required, chain-driven systems offer an alternative that is usually more efficient.

Torque and Power Transmission in Belt-Driven Systems

In belt-driven systems there is usually an input pulley and one or more output pulleys. To determine the maximum torque or power that can be transmitted by the belt, we will need to consider each of the pulleys independently, understanding that slipping occurring at either the input or the output will result in a failure of the power transmission.

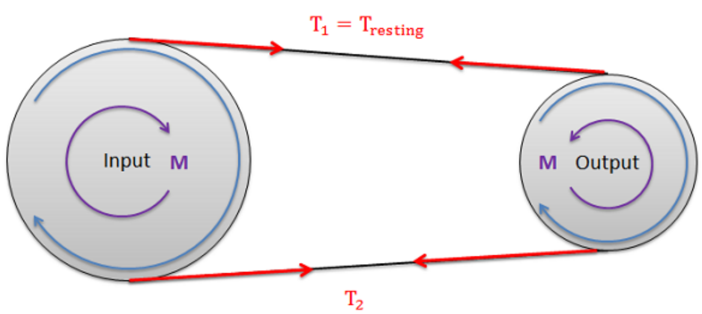


Figure 6.7.8: A belt-driven system with a single input and a single output.

The first step in determining the maximum torque or power that can be transmitted in the belt drive is to determine the maximum possible value for T_2 before slipping occurs at either the input or output pulley (again, slipping at either location cannot occur). To start we will often be given the "resting tension". This is the tension in the belt when everything is stationary and before power is transferred. Sometimes machines will have adjustments to increase or decrease the resting tension by slightly increasing or decreasing the distance between the pulleys. If we turn on the machine and increase the load torque at the output, the tension on one side of the pulleys will remain constant as the resting tension while the tension on the other side will increase. Since the resting tension is constant and is always the lower of the two tensions, it will be the T_1 tension in equations 6.7.1 and 6.7.2.

Though it is often wise to check, assuming the pulleys are made of the same material (and therefore have the same coefficients of friction), it is often assumed that the belt will first slip at the smaller of the two pulleys in a single-input-single-output belt system. This is because the smaller pulley will have the smaller contact angle (β), while all other values remain the same.

Once we have the maximum value for T_2, we can use that to find the torque at the input pulley and the torque at the output pulley. Note that these two values will not be the same unless the pulleys are the same size. To find the torque, we will simply need to find the net moment exerted by the two tension forces, where the radius of the pulley is the moment arm.

Maximum input torque before slipping:

$$M_{max} = (T_{2_{max}} - T_1)(r_{input}) \tag{6.7.4}$$

Maximum output torque before slipping:

$$M_{max} = (T_{2_{max}} - T_1)(r_{output}) \tag{6.7.5}$$

To find the maximum power we can transfer with the belt drive system, we will use the rotational definition of power, where the power is equal to the torque times the angular velocity in radians per second. Unlike the torque, the power at the input and the output will be the same, assuming no inefficiencies.

$$P_{max} = (M_{input\,max})(\omega_{input}) = (M_{output\,max})(\omega_{output}) \tag{6.7.6}$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/nVYc7sBk9UQ.

Example 6.7.1

A steel cable supporting a 60-kg mass is run a quarter of the way around a steel cylinder and supported by a pulling force as shown in the diagram below. The static coefficient of friction between the cable and the steel cylinder is 0.3.

- What is the minimum pulling force required to lift the mass?
- What is the minimum pulling force required to keep the mass from falling?

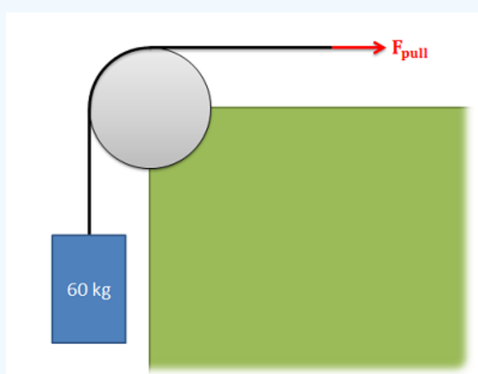


Figure 6.7.9: problem diagram for Example 6.7.1. Belt passing over a flat pulley supports a mass at one end and is supported by a pulling force at the other.

Solution

Video 6.7.2: Worked solution to example problem 6.7.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/SXtKkoF4xtc.

Example 6.7.2

A V-belt pulley as shown below is used to transmit a torque. If the diameter of the pulley below is 5 inches, the resting tension in the belt is 20 lbs, and the coefficient of friction between the belt material and the pulley is 0.4, what is the maximum torque the pulley can exert before slipping?

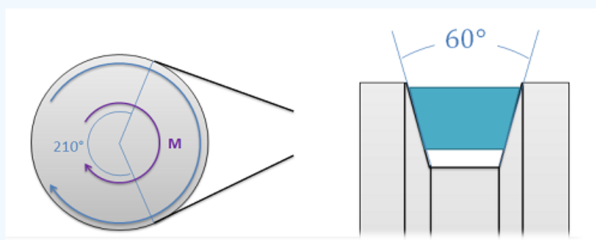


Figure 6.7.10: problem diagram for Example 6.7.2. A V-belt pulley of given dimensions is used to transmit a torque.

Solution

Video 6.7.3: Worked solution to example problem 6.7.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/RLZxKEJVLeo.

Example 6.7.3

A flat belt is being used to transfer power from a motor to an alternator as shown in the diagram below. The coefficient of friction between the belt material and the pulley is 0.5. If we require a power of 100 Watts (Nm/s) while the input is rotating at a rate of 1000 rpm and the output is rotating at a rate of 1428.6 rpm, what is the required resting tension in the belt? (Assume contact angles of approximately 180 degrees)

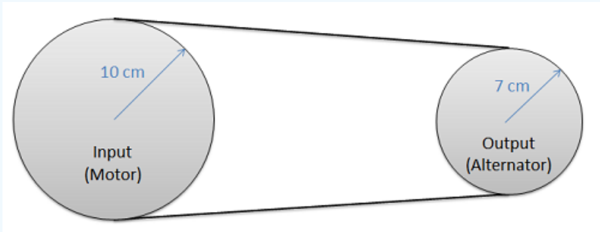


Figure 6.7.11: problem diagram for Example 6.7.3. Belt-driven system of with a single input, single output, and flat belt.

Solution

Video 6.7.4: Worked solution to example problem 6.7.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/K7PhVhXgqUQ.

6.8: Chapter 6 Homework Problems

Exercise 6.8.1

A boy is pulling a sled full of snowballs weighing 30 lbs across a snowy flat surface ($\mu_s = 0.3$, μ_k = 0.1). Find the force F needed to keep the sled moving at a constant speed.

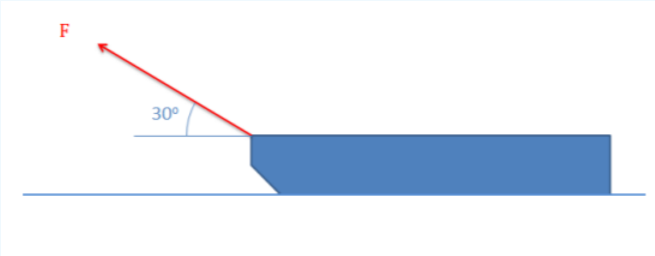


Figure 6.8.1: problem diagram for Exercise 6.8.1. A sled is pulled across a flat surface with a force applied at an angle.

Solution

$F_{pull} = 3.28$ lbs

Exercise 6.8.2

A wooden box sits on a concrete slope ($\mu_s = 0.62$, μ_k = 0.55). How much force would be needed to start pulling this box up the ramp? If we let go of the box, would it slide down the ramp?

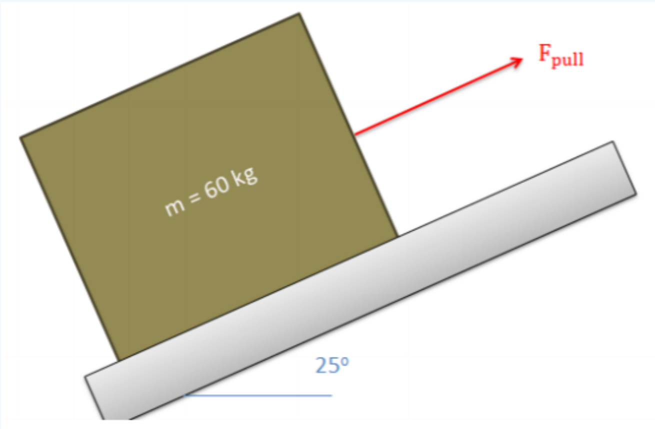


Figure 6.8.2: problem diagram for Exercise 6.8.2. A box is pulled up on a 25° incline by a force applied parallel to the ramp.

Solution

$F_{pull} = 578.9$ N

Box will not slip if released.

Exercise 6.8.3

A wheelbarrow with a weight of 60 lbs and the dimensions shown below sits on a ten-degree incline. Assume friction exists at the rear support (A) but no friction exists at the wheel (B). What is the minimum coefficient of friction needed between the support and the ground to keep the wheelbarrow from sliding down the hill?

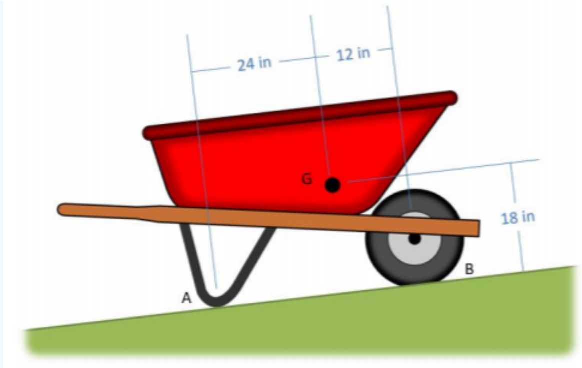


Figure 6.8.3: problem diagram for Exercise 6.8.3. A 60-lb wheelbarrow faces uphill on a 10° incline and experiences friction between the ground and its rear support.

Solution

$\mu_s = 0.418$

Exercise 6.8.4

The car below weighs a total of 1500 lbs, has a center of mass at the location shown, and is rear-wheel drive (only the rear wheels will create a friction force). Assuming that the tires are rubber and the surface is concrete (μ_s = 0.9), what is the maximum angle of the hill (θ) that the car will be able to climb at a constant rate before the wheels start to slip? What is the maximum angle if the car is front-wheel drive?

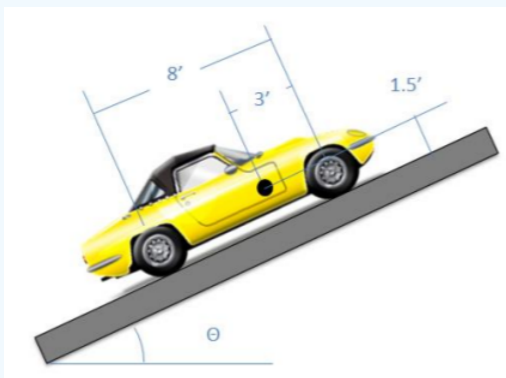


Figure 6.8.4: problem diagram for Exercise 6.8.4. A 1500-lb car climbs uphill at a constant rate on an incline of angle θ.

Solution

$\theta_{max} = 22.0°$ for rear-wheel drive

$\theta_{max} = 25.7°$ for front-wheel drive

Exercise 6.8.5

The fridge shown below has a total weight of 120 lbs and a center of mass as shown below. The fridge is pushed as shown until it either starts to slide or tips over. What is the minimum coefficient of friction needed for the fridge to tip before it starts sliding?

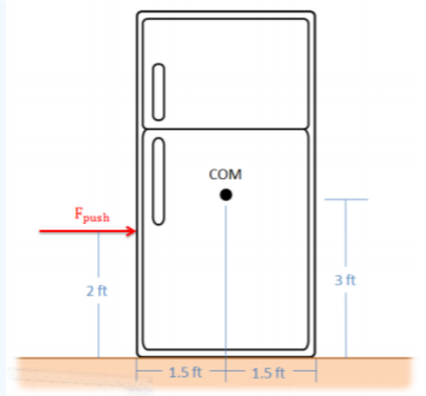


Figure 6.8.5: problem diagram for Exercise 6.8.5. A 120-lb fridge on a level surface experiences a pushing force towards the right.

Solution

$\mu_s = 0.75$ at a minimum

Exercise 6.8.6

You have a bookcase with the dimensions and weight shown below. You are examining the safety of your design.

- If a toddler were to pull on the bookcase as shown, what is the pulling force that would tip it over? (assume the center of gravity is the center of the bookcase and there is no slipping)
- What would the static coefficient of friction need to be to have the case slide before it tips over?

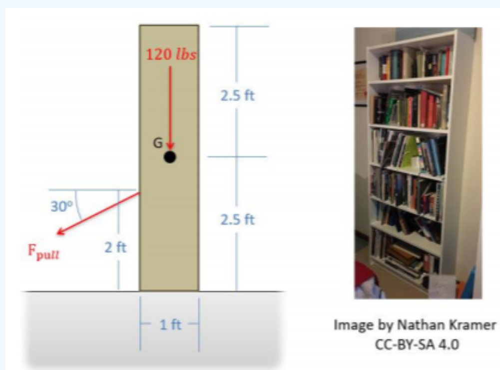


Figure 6.8.6: problem diagram for Exercise 6.8.6. A 120-lb bookcase on a level surface experiences a downwards pulling force applied at an angle on its left side.

Solution

$F_{pull} = 34.64$ lbs

$\mu_s = 0.218$ at a maximum

Exercise 6.8.7

The wedge shown below is pressed by a log splitter into a log with a force of 200 lbs. Assuming the coefficient of friction (both static and kinetic) between the steel wedge and the wood of the log is 0.3, what is the magnitude of the normal force exerted on either side of the log?

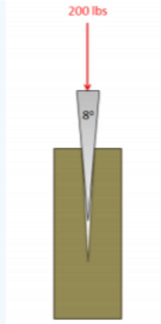


Figure 6.8.7: problem diagram for Exercise 6.8.7. A wedge with a point angle of 8° is pressed point-down into a log.

Solution

$F_{N1} = F_{N2} = 271.0$ lbs

Exercise 6.8.8

The power screw in the screw jack shown below has an outside diameter of one and a half inches and a total of three threads per inch. Assume the coefficients of friction are both 0.16.

- What is the moment required to create a two-ton (4000 lb) lifting force?
- Is this power screw setup self-locking?

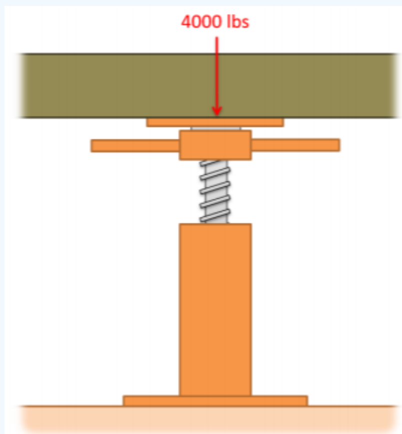


Figure 6.8.8: problem diagram for Exercise 6.8.8. A screw jack experiences a downwards force of 4000 lbs from the load placed on it.

Solution

$M_{lift} = 58.3$ ft-lbs

Screw is self-locking.

Exercise 6.8.9

The end bearing as shown below is used to support a rotating shaft with a load of 300 N on it. If the shaft and the bearing surface are both lubricated steel (assume the coefficients of friction are both 0.06), what is the moment exerted by the friction forces for…

- A solid shaft with a diameter of 2 cm?
- A hollow shaft with an outside diameter of 2 cm and an inside diameter of 1.5 cm?

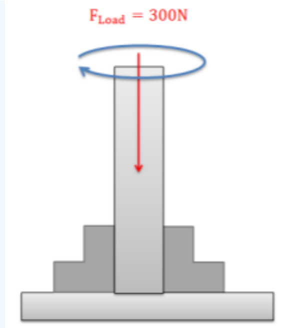


Figure 6.8.9: problem diagram for Exercise 6.8.9. An end bearing supports a rotating shaft that experiences a 300 N load.

Solution

$M_{friction} = 0.12$ N-m (solid shaft)

$M_{friction} = 0.159$ N-m (hollow shaft)

Exercise 6.8.10

A 120-lb person is being lifted by a rope thrown over a tree branch as shown below. If the static coefficient of friction between the rope and the tree branch is 0.61, what is the pulling force required to start lifting the person? What is the pulling force required to keep them from falling?

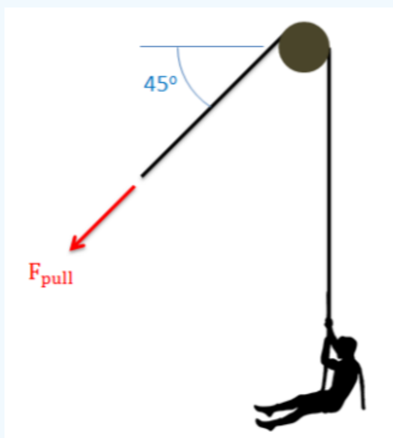


Figure 6.8.10: problem diagram for Exercise 6.8.10. A rope thrown over a branch is pulled downwards at an angle at one end, to lift and hold a 120-lb person on the other end.

Solution

$F_{lift} = 505.1$ lbs

$F_{hold} = 28.5$ lbs

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CHAPTER OVERVIEW

7: Particle Kinematics

7.0: Video Introduction to Chapter 7

Video introduction to the topics covered in this chapter, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/d2TXkfGc-yI.

7.1: One-Dimensional Continuous Motion

Imagine we have a particle that is moving along a single axis. At any given point in time, this particle will have a position, which we can quantify with a single number which we will call x. This value will measure the distance from some set origin point to the position of the particle. If the particle is moving over time, we will need a function to describe the position over time $x(t)$. This is an equation where if we plug a value for t, it will give us the position at that time.

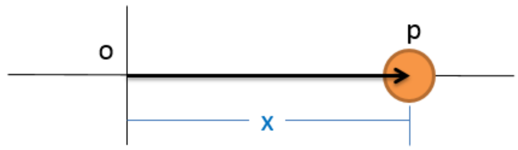

Figure $7.1.1$: The position of a particle in one dimension can be described with a single number. If the position is changing over time, we will use the function $x(t)$ to describe the position at any given point in time.

The velocity of the particle is then the rate of change of the position over time. If the particle is not moving, then position is not changing over time and the velocity is zero. If the particle is moving, we will first need to find the equation for position $x(t)$, and then take the derivative of the position equation to find the velocity equation $v(t)$. Velocity differs from speed in that the velocity has a direction (either positive or negative for now) while the speed is simply the magnitude of the velocity (always a positive number).

Next up is the acceleration, which is the rate of change of the velocity over time. If the velocity is not changing, the acceleration will be zero. If the velocity does change over time, then we will need to take the derivative of the velocity equation $v(t)$ to find the acceleration equation $a(t)$. The acceleration is then also the double derivative of the position equation over time. Like the velocity, the acceleration has both a magnitude and a direction.

To simplify the notation, we often use a dot on top of the variable to indicate a time derivative. This makes the velocity (the derivative of x) $\dot{x}$ and the acceleration (the derivative of the derivative of x) $\ddot{x}$. These relationships and their shorthand notations are all shown below.

$$\text{Position:} \quad x(t) \tag{7.1.1}$$

$$\text{Velocity:} \quad v(t) = \frac{dx}{dt} = \dot{x} \tag{7.1.2}$$

$$\text{Acceleration:} \quad a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x} \tag{7.1.3}$$

If we instead start with the equation for acceleration, we can take the integral of that equation $a(t)$ to find the equation for velocity, $v(t)$. But unlike the derivatives, we will have an extra step in this process because whenever we integrate we wind up with a constant of integration (which we will usually call C). When we integrate the acceleration equation to find the velocity equation, this constant will be the initial velocity (the velocity at time = 0).

Next we can take the integral of the velocity equation $v(t)$ to find the position equation $x(t)$. With this integration we will again wind up with a constant of integration, which in this case will be the initial position (the position at time = 0). These relationships are shown below.

$$\text{Acceleration:} \quad a(t) \tag{7.1.4}$$

$$\text{Velocity:} \quad v(t) = \int a(t) \tag{7.1.5}$$

$$\text{Position:} \quad x(t) = \int v(t) = \iint a(t) \tag{7.1.6}$$

Constant Acceleration Systems:

In cases where we have a constant acceleration (often due to a constant force), we can start with a constant value for $a(t) = a$, and work out the integrals from there. Along the way we will add the initial velocity and the initial position as the constants of integration to wind up with the formulas below.

$$\begin{aligned}\text{Acceleration:}\quad & a(t)=a & (7.1.7)\\ \text{Velocity:}\quad & v(t)=at+v_0 & (7.1.8)\\ \text{Position:}\quad & x(t)=\frac{1}{2}at^2+v_0t+x_0 & (7.1.9)\end{aligned}$$

If we take the equations for the position and the velocity from above, then solve both of them for t and set those equations equal to one another, we can actually wind up with another equation that directly relates position, velocity, and acceleration without needing to know the time.

$$v^2-v_0^2=2a(x-x_0) \tag{7.1.10}$$

It is important to remember that these equations are only valid when the acceleration is constant. When that is not the case, you will need to use calculus to find the derivatives or integrals based on the equations for position, velocity, and acceleration that you do know.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/uemxYyLN3-U.

Example 7.1.1

You are in a van that steadily accelerates from 20 m/s to 35 m/s over the course of 10 seconds. What is your rate of acceleration?

Figure 7.1.2: A black van accelerating at a constant rate.

Solution

Video 7.1.2: Worked solution to example problem 7.1.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/19qGXfNkWcQ.

? Example 7.1.2

You are in a van that steadily accelerates from 20 m/s to 35 m/s over the course of 10 seconds. How many meters did you travel in those ten seconds?

Figure 7.1.3: A black van accelerating at a constant rate.

Solution

Video 7.1.3: Worked solution to Example 7.1.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/hjyUHnw8FSI.

❓ Example 7.1.3

In a rocket sled deceleration experiment, a manned sled is decelerated from a speed of 200 mph (89.4 m/s) to a stop at a constant rate of 18 G's (176.6 m/s^2). How long does it take for the sled to stop? How far does the sled travel while decelerating?

Figure 7.1.4: A rocket sled decelerating at a constant rate.

Solution

Video 7.1.4: Worked solution to Example 7.1.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/bXzcdpaxB_c.

Example 7.1.4

A metallic particle is accelerated in a magnetic field such that its velocity over time is defined by the function $v(t) = 4t^2 - 12$, where time is in seconds and velocity is in meters per second. If we assume that the particle has an initial position of zero $(x_0 = 0)$, what are the equations that describe the acceleration and position over time?

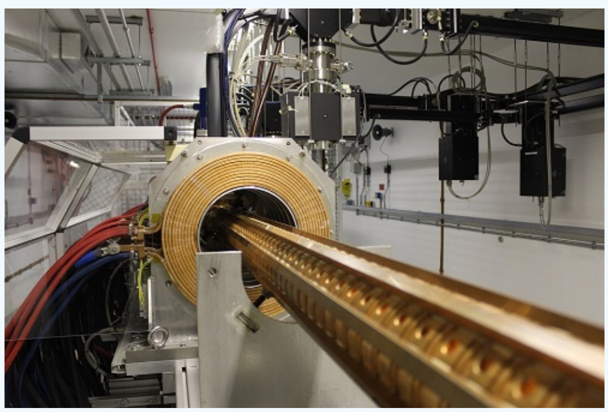

Figure 7.1.5: A particle accelerator.

Solution

Video 7.1.5: Worked solution to Example 7.1.4, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/AWhLmMhKygg.

7.2: One-Dimensional Noncontinuous Motion

In continuous motion, we used a single mathematical function each to describe the position, velocity, or acceleration over time. If we cannot describe the motion with a single mathematical function over the entire time period, that motion is considered **noncontinuous** motion. In cases such as this, we will use different equations for different sections of the overall time period.

For an example of noncontinuous motion, imagine a car that accelerates for a few seconds, then holds a steady speed for a few seconds, then puts on the brakes and comes to a stop over the final few seconds. There is no one mathematical function we can use to describe the motion for the full time period, but if we break the motion into three pieces, then we can come up with an equation for each section of the motion.

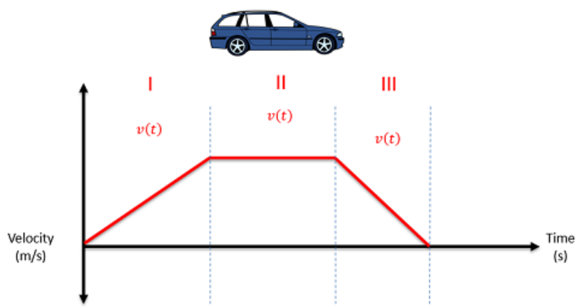

Figure $7.2.1$: As this car accelerates, its velocity steadily goes up for a few seconds, then it holds constant for a few seconds, then it steadily goes down for a few seconds. Since we would need a separate equation for each section of this motion, this is considered noncontinuous motion.

Analyzing the first time period will be exactly the same as analyzing a continuous function. We will initially need to identify the mathematical function to describe position, or velocity, or acceleration for that first time period. Next we take derivatives to move from position to velocity to acceleration or take integrals to move from acceleration to velocity to position. Whenever we take an integral, we need to remember to include the constant of integration which will represent the initial velocity or the initial position (in the velocity and position equations, respectively).

For the second, third, and any other following sections, we will do much the same process. We will start by identifying an equation for the position, or velocity, or acceleration for that time period. From there we again take derivatives or integrals as appropriate, but now the constants of integration will be a little more complicated. Those constants still represent initial velocities and and positions in a sense, but they will be the velocity and position when $t = 0$, not the velocity and position at the start of that section.

To find the constants of integration, we are instead going to have to use the transition point, which is the point in time where we are moving from one set of equations to the next. Even though the equations are changing, we cannot have an instantaneous jump in either the position or velocity. An instantaneous jump in either position or velocity would require infinite acceleration, which is physically impossible.

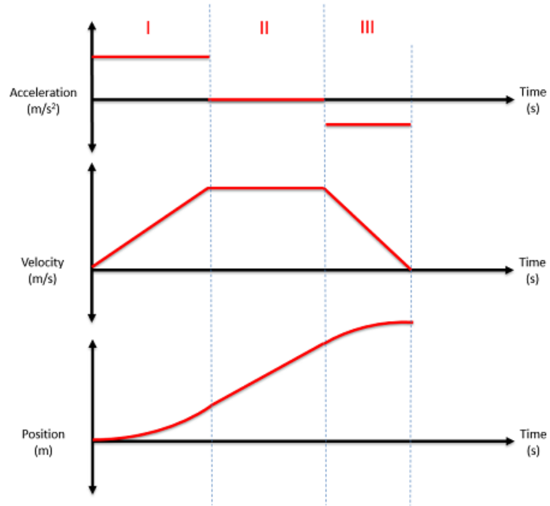

Figure $7.2.1$. Though there are noticeable jumps in the acceleration moving from one section to the next, there are no jumps in the velocity or position as we move from one section to the next. This is because jumps in those plots would require infinite accelerations.

To find the velocity equations for the second time frame (or third, fourth, etc.), we start by integrating the acceleration equation for that same time period. This will lead to an equation with an unknown constant of integration. To solve for that constant, we look back to the velocity equation for the previous time frame and solve for the velocity at the very end of this prior time period. Since it can't jump instantaneously, this is also the velocity at the start of the current time period. Using this velocity, along with the time t at the transition point, we can solve for the last unknown in the current velocity equation (the constant of integration).

To find the position equation for the second time frame (or third, fourth, etc.), we start by integrating the velocity equation for the same time period (you will need to solve for the unknowns in the velocity equation first, as discussed above). After integration, we should have one new constant of integration in the position equation. Just as we did with the velocity equations, we will use the position equation from the prior time frame to solve for the position at the transition point, then use that value along with the known time t to solve for the unknown constant in the current position equation.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/yNBenQRalhU.

Example 7.2.1

A car accelerates from rest at a rate of 10 m/s^2 for 10 seconds. The car then immediately begins decelerating at a rate of 4 m/s^2 for another 25 seconds before coming to a stop. Find the equations for the acceleration, velocity, and position functions over the full 35-second time period, and plot these functions.

Figure 7.2.3: A red racecar moving along a track.

Solution

Video 7.2.2: Worked solution to example problem 7.2.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/tPBlDQsiX_c.

? Example 7.2.2

A plane with an initial speed of 95 m/s touches down on a runway. For the first second the plane rolls without decelerating. For the next 5 seconds reverse thrust is applied, decelerating the plane at a rate of 4 m/s^2. Finally, the brakes are applied with reverse thrust increasing the rate of deceleration to 8 m/s^2. How long does it take for the plane to come to a complete stop? How far does the plane travel before coming to a complete stop?

Figure 7.2.4: A plane is moving down an airport runway.

Solution

Video 7.2.3: Worked solution to example problem 7.2.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/Of-ipYWblrQ.

? Example 7.2.3

A satellite's motion is described by the velocity function shown below over a sixty-second time period. For that same time period, determine the satellite's acceleration and position functions and draw these functions on a plot.

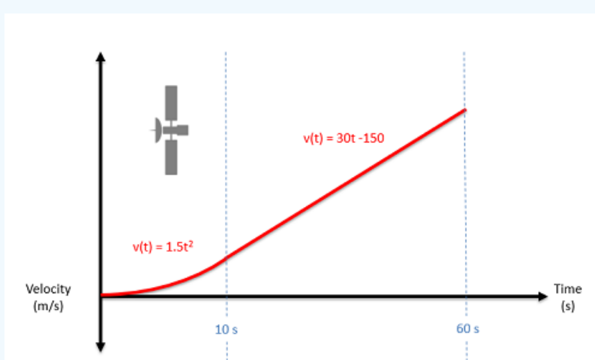

Figure 7.2.5: problem diagram for Example 7.2.3. Graph of a satellite's velocity in m/s for a period of 60 seconds, with the functions describing the velocity given.

Solution

Video 7.2.4: Worked solution to example problem 7.2.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/ngI6_I8VFPs.

7.3: Two-Dimensional Kinematics with Rectangular Coordinates

Two-dimensional motion (also called planar motion) is any motion in which the objects being analyzed stay in a single plane. When analyzing such motion, we must first decide the type of coordinate system we wish to use. The most common options in engineering are rectangular coordinate systems, normal-tangential coordinate systems, and polar coordinate systems. Any planar motion can potentially be described with any of the three systems, though each choice has potential advantages and disadvantages.

The **rectangular coordinate system** (also sometimes called the Cartesian coordinate system) is the most intuitive approach to describing motion. In rectangular coordinate systems we have an x-axis and a y-axis. These axes remain fixed to some origin point in the environment and they do not change over time. Instead, the bodies we are analyzing usually move relative to these fixed axes. An example of a body with a rectangular coordinate systems is shown in the figure below.

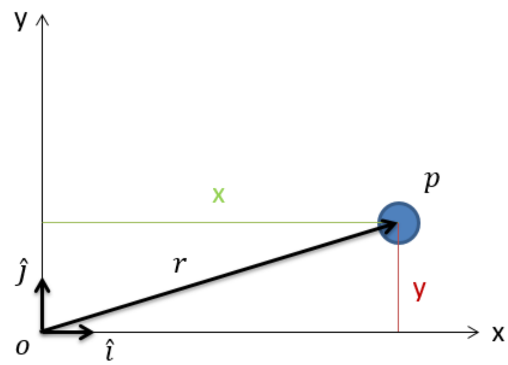

Figure 7.3.1: In the rectangular coordinate system we have a fixed origin point at o, the particle at point p, and the x and y directions, which must be perpendicular to one another. The vector $\vec{r}$ is the vector going from o to p. The component of this vector in the x direction is x and the component of this vector in the y direction is y. We usually describe position in terms of x and y at any given point in time. The vectors $\hat{\imath}$ and $\hat{\jmath}$ represent unit vectors (vectors with a length of one) in the x and y directions respectively.

Rectangular coordinates work best for systems where **all forces maintain a constant direction**. The most common example of this is projectile motion, where gravity (the only force in these systems) maintains a constant downward direction. An example of a system where the forces change direction over time would be something like a car going around a curve in the road. In this case, the friction force at the tires is going to be rotating with the car. The car problem will therefore be better suited to the use of normal-tangential or polar coordinate systems.

When describing the position of a point in rectangular coordinate systems, we are going to start by describing both x and y coordinates in a vector form. For this, the values x and y represent distances and the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are used to indicate which distance corresponds with which direction. This may seem redundant, but remember when solving actual problems, x and y will just be numbers.

$$\text{Position:} \quad r_{p/o}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} \tag{7.3.1}$$

Just like with one-dimensional problems, if we take the derivative of the position equation, we will find the velocity equation. If we take the derivative of the velocity equation we will wind up with the acceleration equation. Also like one-dimensional problems, we can use integration to move in the other direction, moving from an acceleration equation to a velocity equation to a position equation.

The unit vectors add a new element in two dimensions, but since the unit vectors don't change over time (they are constants), we treat them like we would any other constant for derivatives and integrals. The resulting velocity and acceleration equations are as follows.

$$\text{Velocity:} \quad v(t) = \dot{x}(t)\,\hat{\imath} + \dot{y}(t)\,\hat{\jmath} \tag{7.3.2}$$

$$\text{Acceleration:} \quad a(t) = \ddot{x}(t)\,\hat{\imath} + \ddot{y}(t)\,\hat{\jmath} \tag{7.3.3}$$

The above equations are vector equations with velocities and accelerations broken down into x and y components. Since the x and y directions are perpendicular, they are also independent (movement in the x direction doesn't impact movement in the y direction, and vice versa). This essentially means we can split our vector equation into a set of two scalar equations. To do this we just put everything in front of the $\hat{i}$ unit vectors in the x equations and everything in front of the $\hat{j}$ unit vectors in the y equations.

$$\text{Position:} \quad x(t) = \ldots \qquad y(t) = \ldots \tag{7.3.4}$$

$$\text{Velocity:} \quad \dot{x}(t) = \ldots \qquad \dot{y}(t) = \ldots \tag{7.3.5}$$

$$\text{Acceleration:} \quad \ddot{x}(t) = \ldots \qquad \ddot{y}(t) = \ldots \tag{7.3.6}$$

Once we have everything split into x- and y-direction equations, we can just use the same processes we used for one dimensional motion to move from x to $\dot{x}$ to $\ddot{x}$, and from y to $\dot{y}$ to $\ddot{y}$. The variable linking the two equations is time (t).

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/qKqUNuSPWT4.

Example 7.3.1

A motorcycle drives off a one-meter-tall ramp at an angle of 30 degrees as shown below. Determine the equations for the acceleration, velocity, and position over time. How far does the motorcycle travel in the x direction before hitting the ground?

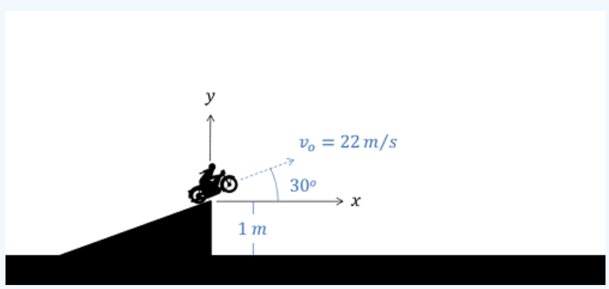

Figure 7.3.2: problem diagram for Example 7.3.1. A motorcycle drives off the edge of a ramp with an initial velocity of 22 m/s at 30° above the x-axis.

Solution

Video 7.3.2: Worked solution to example problem 7.3.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/ujM4LIQlG1Q.

? Example 7.3.2

A basketball is thrown towards a hoop that is three feet higher in the y direction and 25 feet away in the x direction. If the ball is thrown at an initial angle of 50 degrees, what must the initial velocity be for the ball to make it into the hoop?

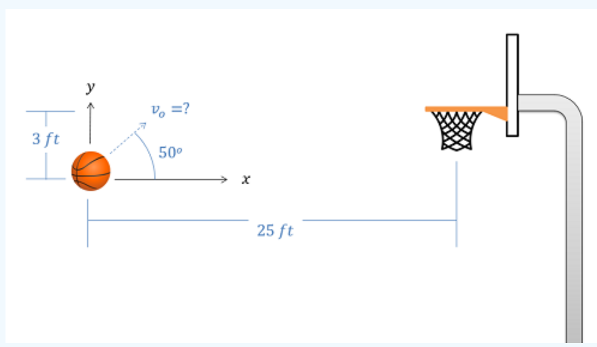

Figure 7.3.3: problem diagram for Example 7.3.2. A ball is thrown at some initial velocity, at an angle of 50° above the horizontal.

Solution

Video 7.3.3: Worked solution to example problem 7.3.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/IGBB1YXRNr0.

7.4: Two-Dimensional Kinematics with Normal-Tangential Coordinates

Two-dimensional motion (also called planar motion) is any motion in which the objects being analyzed stay in a single plane. When analyzing such motion, we must first decide the type of coordinate system we wish to use. The most common options in engineering are rectangular coordinate systems, normal-tangential coordinate systems, and polar coordinate systems. Any planar motion can potentially be described with any of the three systems, though each choice has potential advantages and disadvantages.

The **normal-tangential coordinate system** centers on the body in motion. The origin point will be the body itself, meaning that the position of the particle in the normal-tangential coordinate system is always "zero". The tangential direction (t-direction) is defined as the direction of travel at that moment in time (the direction of the current velocity vector), with the normal direction (n-direction) being 90 degrees counterclockwise from the t-direction. The diagram below shows a particle following a curved path with the current normal and tangential directions.

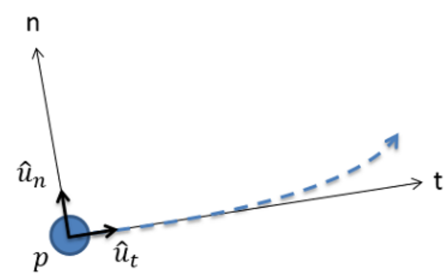

Figure 7.4.1: In the normal-tangential coordinate system, the particle itself serves as the origin point. The t-direction is the current direction of travel and the n-direction is always 90° counterclockwise from the t-direction. The $\hat{u}_t$ and $\hat{u}_n$ vectors represent unit vectors in the t and n directions respectively.

Normal-tangential coordinate systems work best when we are observing motion from the perspective of the body in motion, such as being a passenger in a car or plane. In such cases, we would define ourself as the origin point and "forward" would be the tangential direction. An important distinction between the rectangular coordinate system and the normal-tangential coordinate system is that the axes are not fixed in the normal-tangential coordinate system. If we go back to the car example, the "forward" or tangential direction will turn with the car, but the "east" direction or the x-direction will remain constant no matter which way the car is pointed.

The way the coordinate system is defined, the position of the particle is always set to be at the origin point. The velocity is also always set to be in the tangential direction, and thus there is no velocity in the n-direction. The variable v is the body's current speed.

$$\text{Position:} \quad r_{p/o}(t) = 0\,\hat{u}_t + 0\,\hat{u}_n \tag{7.4.1}$$

$$\text{Velocity:} \quad v(t) = v\,\hat{u}_t \tag{7.4.2}$$

To find the acceleration, we need to take the derivative of the velocity function. This may seem simple, but there is a new thing to consider in that the u_t unit vector is not constant. This means a change in speed can cause an acceleration, and a change in direction (which would change the u_t direction) can also cause an acceleration. Going back to our car example, this makes some intuitive sense. We can feel accelerations, and we would be able to feel acceleration if we suddenly stepped on the gas and increased our speed, but we would also be able to feel the acceleration if we took a tight turn at a constant speed.

Going back to the derivative, we will use the product rule, taking the derivative of one piece at a time.

$$\text{Acceleration:} \quad a(t) = \dot{v}\hat{u}_t + v\dot{\hat{u}}_t \tag{7.4.3}$$

$\dot{v}$ is the rate of change of speed of the body, which is called the tangential acceleration. Going back to our car analogy, this is the acceleration we would experience from pressing the gas or brake pedals.

The other piece of our derivative is the speed times the derivative of a unit vector, which we will need to analyze further. When thinking about the derivative of a rotating unit vector, we think about rotating the coordinate system by a small amount $d\theta$.

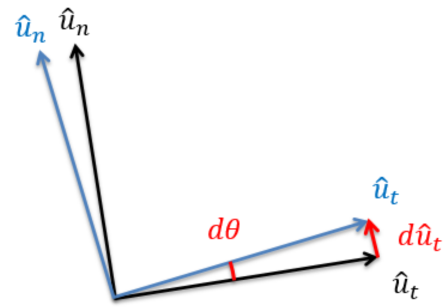

Figure 7.4.2: The derivative of a rotating unit vector can be thought of as the change in position of the head of that vector as it rotates about a small angle $d\theta$.

The derivative of the $\hat{u}_t$ vector is then the change in position of the head of the vector. Using some geometry, we can see that the distance the head of the vector moves is the length of the vector (which is always 1 for a unit vector) times the angle of rotation in radians. The direction the head of the $\hat{u}_t$ vector travels is roughly the $\hat{u}_n$ direction. In fact, as $d\theta$ approaches zero, it becomes exactly the $\hat{u}_n$ direction. Putting this back into our derivative, we wind up with the following equation for acceleration.

$$a(t) = \dot{v}\hat{u}_t + v\dot{\hat{u}}_t = \dot{v}\hat{u}_t + v\dot{\theta}\hat{u}_n \tag{7.4.4}$$

Before we arrive at our final set of equations, we have one last potential substitution. If a particle is moving along a curved path, the rate at which it is turning $(\dot{\theta})$ will be equal to the velocity of the particle divided by the radius of the path at that point (v divided by rho, ρ). Putting this last substitution in, we have our final set of equations with two equivalent options for calculating the accelerations.

$$a(t) = \dot{v}\hat{u}_t + v\dot{\theta}\hat{u}_n = \dot{v}\hat{u}_t + \frac{v^2}{\rho}\hat{u}_n \tag{7.4.5}$$

When using these equations, it is important to remember that they are acceleration equations. If we want to know the overall acceleration we would need to add the two acceleration components **as vectors**. Also, if we are given an acceleration that is not in the normal or tangential direction, we will first need to break that acceleration vector down into normal and tangential components before using the above equations. Finally, if we want the velocity or acceleration in directions other than the normal and tangential directions, we will need to use a coordinate transformation.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/TIjJLcmtaq8.

Example 7.4.1

A commercial jetliner is traveling at a constant 250 m/s when it executes an emergency 180-degree turn. If the turn takes 20 seconds, what is the acceleration experienced by the passengers? What is the radius of the curve taken by the plane?

Solution

Video 7.4.2: Worked solution to example problem 7.4.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/1cQ-LQC8Ahg.

Example 7.4.2

Assuming a cloverleaf interchange has a radius of curvature of 80 meters at the tightest part of the turn, what is the fastest a car could travel around this curve without experiencing more than half a g in acceleration? Assume the car is traveling at a constant speed. If the car was instead increasing speed at a rate of 2 m/s^2, what would be the new overall magnitude of the acceleration experienced by the passengers?

Figure 7.4.3: A cloverleaf interchange on a highway.

Solution

Video 7.4.3: Worked solution to example problem 7.4.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/_Df_HZU0yHk.

7.5: Two-Dimensional Motion with Polar Coordinates

Two-dimensional motion (also called planar motion) is any motion in which the objects being analyzed stay in a single plane. When analyzing such motion, we must first decide the type of coordinate system we wish to use. The most common options in engineering are rectangular coordinate systems, normal-tangential coordinate systems, and polar coordinate systems. Any planar motion can potentially be described with any of the three systems, though each choice has potential advantages and disadvantages.

The **polar coordinate system** uses a distance (r) and an angle (θ) to locate a particle in space. The origin point will be a fixed point in space, but the r-axis of the coordinate system will rotate so that it is always pointed towards the body in the system. The variable r is also used to indicate the distance from the origin point to the particle. The theta-axis will then be 90 degrees counterclockwise from the r-axis with the variable θ being used to show the angle between the r-axis and some fixed axis that does not rotate. The diagram below shows a particle with a polar coordinate system.

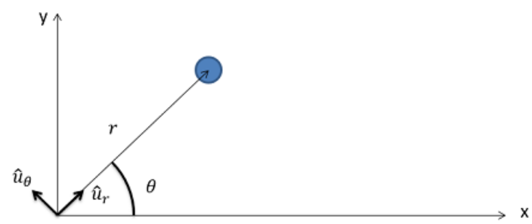

Figure 7.5.1: In the polar coordinate system, the r direction always points from the origin point to the body. The variable r is also used to indicate the distance between the two points. The theta direction will always be 90° counterclockwise from the r direction. Theta is also used to indicate the angle between the r direction and some fixed axis used for reference. The $\hat{u}_r$ and $\hat{u}_\theta$ vectors represent unit vectors in the r and θ directions, respectively.

Polar coordinate systems work best in systems where a body is being tracked via a distance and an angle, such as a radar system tracking a plane. In cases such as this, the raw data from this in the form of an angle and a distance would be direct measures of θ and r respectively. Polar coordinate systems will also serve as the base for extended body motion, where motors and actuators can directly control things like r and θ.

The way the coordinate system is defined, the r-axis will always point from the origin point to the body. The distance from the origin to the point is defined as r with no component of the position being in the θ direction.

$$\text{Position:} \quad r_{p/o}(t) = r\hat{u}_r + 0\ \hat{u}_\theta \tag{7.5.1}$$

To find the velocity, we need to take the derivative of the position function over time. Since the distance r can change over time as well as the direction $\hat{u}_r$ changing over time to track the body, we need to worry about the derivative of r as well as the derivative of the unit vector. Like we did with the normal-tangential systems, we will use the product rule and then substitute in a value for the derivative of the unit vector.

$$\text{Velocity:} \quad v(t) = \dot{r}\hat{u}_r + r\dot{\hat{u}}_r = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta \tag{7.5.2}$$

To find the acceleration, we need to take the derivative of the velocity function. As all of these terms, including the unit vectors, change over time, we will need to use the product rule extensively. The $\hat{u}_r$ term will split into two terms, and the $\hat{u}_\theta$ term will split into three terms.

$$\text{Acceleration:} \quad a(t) = \ddot{r}\hat{u}_r + \dot{r}\dot{\hat{u}}_r + \dot{r}\dot{\theta}\hat{u}_\theta + r\ddot{\theta}\hat{u}_\theta + r\dot{\theta}\dot{\hat{u}}_\theta \tag{7.5.3}$$

Again we will need to substitute in values for the derivatives of the unit vectors similar to before, but it is worth mentioning that the derivative of the $\hat{u}_\theta$ vector as it rotates counterclockwise is in the negative $\hat{u}_r$ direction.

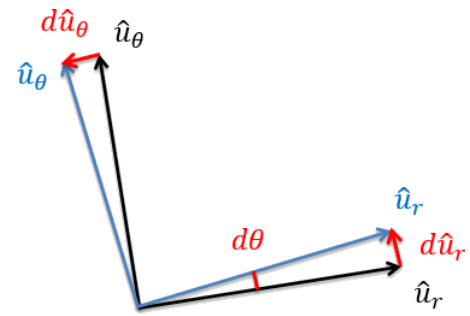

Figure 7.5.2: The derivatives of the $\hat{u}_r$ and $\hat{u}_\theta$ unit vectors. Notice that the derivative of the $\hat{u}_\theta$ vector is in the negative $\hat{u}_r$ direction.

After substituting in the derivatives of the unit vectors and simplifying the function, we arrive at our final equation for the acceleration.

$$\text{Acceleration:}\quad a(t) = \ddot{r}\hat{u}_r + \dot{r}\dot{\theta}\hat{u}_\theta + \dot{r}\dot{\theta}\hat{u}_\theta + r\ddot{\theta}\hat{u}_\theta - r\dot{\theta}^2\hat{u}_r \tag{7.5.4}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{u}_\theta \tag{7.5.5}$$

Though this final equation has a number of terms, it is still just two components in vector form. Just as with the normal-tangential coordinate system, we will need to remember that we will need to split the single vector equation into two separate scalar equations. In this case we will have the equation for the terms in the r direction and the equation for the terms in the θ direction.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/F4i0Kz660aE.

? Example 7.5.1

A radar tracking station gives the following raw data to a user at a given point in time. Based on this data, what is the current velocity and acceleration in the r and θ directions? What is the current velocity and acceleration in the x and y directions?

$\theta = 40°$	$r = 6400$ ft
$\dot{\theta} = -.039$ rad/s	$\dot{r} = 312$ ft/s
$\ddot{\theta} = .003807$ rad/s^2	$\ddot{r} = 9.751$ ft/s^2

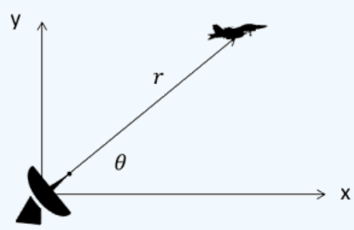

Figure 7.5.3: Problem diagram for Example 7.5.1. The instantaneous polar-coordinate position values of an airplane are given as tracked by a radar station, as well as the first and second derivatives of these quantities.

Solution

Video $7.5.2$: Worked solution to example problem $7.5.1$, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/HP4WiIa3Nc0.

? Example $7.5.2$

A spotlight is tracking an actor as he moves across the stage. If the actor is moving with a constant velocity as shown below, what values do we need for the spotlight angular velocity $(\dot{\theta})$ and spotlight angular acceleration $(\ddot{\theta})$ so that the spotlight remains fixed on the actor?

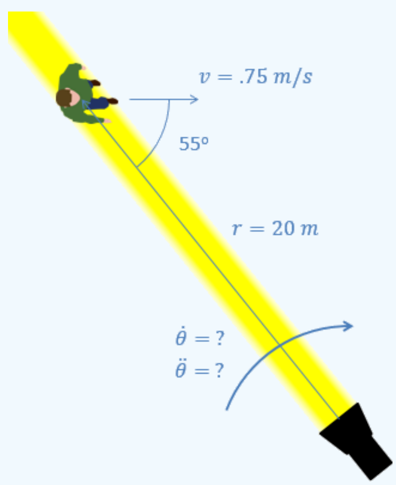

Figure $7.5.4$: problem diagram for Example $7.5.2$. A spotlight rotates to follow an actor moving across stage at a known velocity and starting from a known position in relation to the spotlight.

Solution

Video 7.5.3: Worked solution to example problem 7.5.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/jyD1seNQI14.

7.6: Dependent Motion Systems

Dependent motion analysis is used when two or more particles have motions that are in some way connected to one another. The way in which the motion of these particles is connected is known as the constraint. A simple example of a constrained system is shown in the figure below. Imagine that someone's pickup truck gets stuck in the sand, and a friend uses a rope to help pull it out. This friend ties one end of the rope to the rear bumper of her car, loops the rope around a bar on the front of the pickup truck, then ties the other end to a stationary tree. In this case the two vehicles will not have the same velocity or acceleration, but their motions are related because they are tied together by the rope. In this case, the rope is acting as the constraint, allowing us to know the velocity or acceleration of one vehicle based on the velocity or acceleration of the other vehicle.

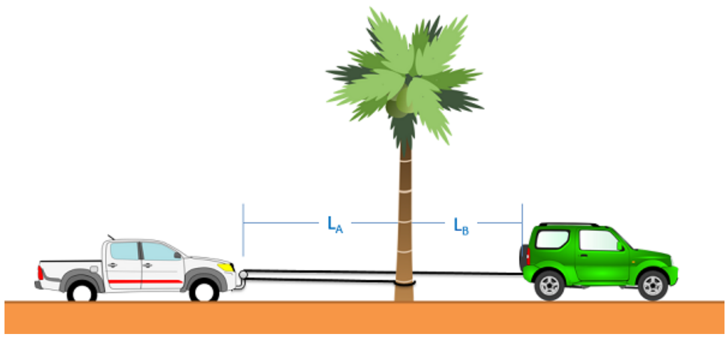

Figure 7.6.1: This represents a constrained system. The motion of the car and the pickup truck will be related to one another via rope that is connecting them.

The first thing we will need to do when analyzing these systems is to come up with what is known as the **constraint equation**. A constraint equation will be some geometric relationship that will remain true over the course of the motion. In the above example, imagine the rope is 50 feet long. Using the tree as a stationary point, we can also say that the length of the rope is the distance from the green car to the tree, plus two times the distance from the pickup truck to the tree (since it must go out to the pickup truck and then back). If we put this into an equation (the constraint equation) we would have the following.

$$\text{Constraint Equation: Positions} \qquad L = 50\ ft = 2L_A + L_B \tag{7.6.1}$$

Once we have a constraint equation that works for positions, we can take the derivative of this equation to find another constraint equation that relates velocities. In this case, the length of the rope is constant, and therefore the derivative of the length will be zero. If we take the derivative of the constraint equation again, we wind up with a third constraint equation that relates accelerations.

$$\text{Constraint Equation: Velocities} \qquad \dot{L} = 0 = 2\dot{L}_A + \dot{L}_B \tag{7.6.2}$$

$$\text{Constraint Equation: Accelerations} \qquad \ddot{L} = 0 = 2\ddot{L}_A + \ddot{L}_B \tag{7.6.3}$$

In these equations, it is important to remember that the values represent the changes in length, rather than direct measures of velocities. Though both vehicles would have positive velocities in the example above (velocities to the right), one $\dot{L}$ value will be positive and one will be negative. This is because the truck is getting closer to the tree while the car is getting further away. A similar situation will occur for the accelerations, where both vehicles would have positive accelerations even if the $\ddot{L}$ values are a mix of positive and negative values.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/J1aPSrdDzdk.

? Example 7.6.1

A truck becomes stuck in the sand at a local beach. To help, a friend takes a rope 50 feet in length, ties one end to her car, loops the rope around a bar at the front of the truck, and then ties the other end to a stationary tree as shown below. If the car accelerates at a rate of 0.2 ft/s², what will the velocity of the truck be by the time it gets to the tree?

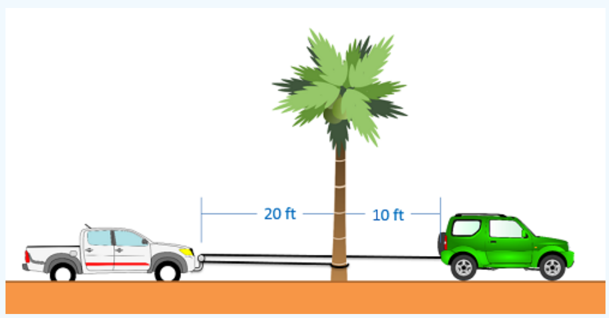

Figure 7.6.2: problem diagram for Example 7.6.1. A rope looped around the front of a truck, tied at one end to a tree trunk and at the other to a car's rear bumper, allows the car to pull the truck free of the sand it is stuck in.

Solution

Video 7.6.2: Worked solution to example problem 7.6.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/IxO_Nrs7Kj0.

? Example 7.6.2

A man has hooked up a pulley, a rope, and a platform as shown below to lift loads up onto a nearby rooftop. If x is currently 15 meters, y is currently 5 meters, and the man is walking away from the building at a rate of 0.5 meters per second, what is the current velocity of the platform?

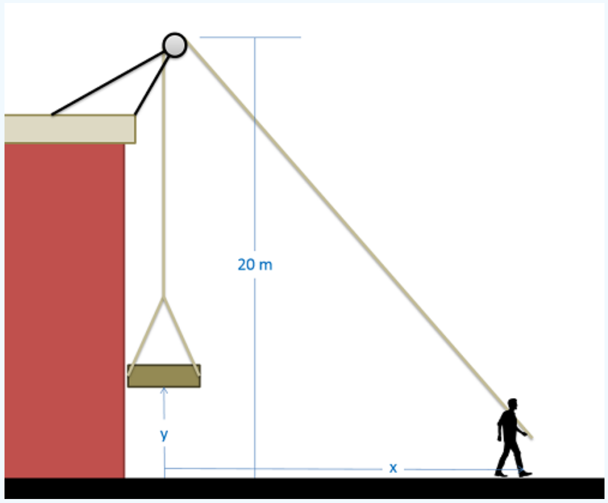

Figure 7.6.3: problem diagram for Example 7.6.2. A man walks away from a 20-meter-high building, holding one end of a rope that passes over a pulley on the rooftop and raises a load on its other end.

Solution

Video 7.6.3: Worked solution to example problem 7.6.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/z3D_2jHLCik.

7.7: Relative Motion Systems

Relative motion analysis is the analysis of bodies where more than one body is in motion and we are in some way examining the motion of one moving particle relative to another moving particle. An example would be cars on a highway. If you are in one of many cars moving down one side of the highway, you and the other cars will all have velocities in a single direction. From your perspective inside the car, though, fast cars in front of you will appear to be moving away from you and slow cars in front of you will appear to be moving towards you. By knowing your speed, and by observing how fast the cars are moving away from you or towards you, you could determine or at least estimate how fast these cars are going. The process of using your speed in conjunction with the relative velocities of the other cars to find the other cars' speed is relative motion analysis.

Relative Motion in One Dimension:

In a single dimension, we will usually have at least two moving particles as well as some fixed reference point. We usually call the fixed reference point O, and then label the other points A, B, and so on. In the diagram, below we can see an example of this, with a fixed reference point at O, a police car at A, and a speeding car at B.

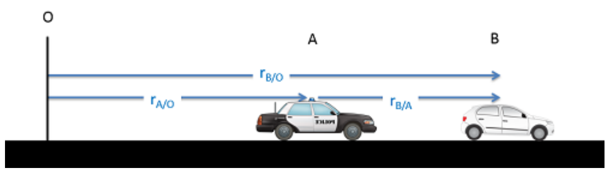

Figure 7.7.1: In a single dimension we have a fixed reference point O, as well as at least two moving bodies (labeled as A and B in this case). The r vectors represent the distances between bodies, with the subscript indicating the point we are observing and the point we are observing from. The subscripts should always have the following format: point being observed / point we are observing from.

As we can see from the diagram above, the sum of the distance from O to A $(r_{A/O})$ and the distance from A to B $(r_{B/A})$ will be equal to the distance from O to B $(r_{B/O})$. This gives us our position equation below.

$$\text{Position:} \quad r_{B/O} = r_{A/O} + r_{B/A} \tag{7.7.1}$$

One thing we should notice about the subscripts is that if we keep to our standard naming convention (point being observed / point we are observing from) then we should be able to see which points cancel out and which remain. In this case, the A's on the top and bottom on the right cancel out, leaving just a B on the top and an O on the bottom. This matches the subscript on the left of the equation.

Furthermore, we can take the derivative of this equation to relate velocities, or a double derivative to relate accelerations. This means that if we know any two of the terms in the equations below, we can solve for the third.

$$\text{Velocity:} \quad v_{B/O} = v_{A/O} + v_{B/A} \tag{7.7.2}$$

$$\text{Acceleration:} \quad a_{B/O} = a_{A/O} + a_{B/A} \tag{7.7.3}$$

Relative Motion in Two Dimension:

Just as with one dimension, we can start by examining a set of relative positions. In this case we will use the example of two planes moving through the sky. We may wish to deduce the position of plane B on a map, based on our observations from plane A. By using the information we have on our position relative to some set location, along with the relative position read in from on board instrumentation, we should be able to determine the absolute position (including an x and y coordinate) for plane B.

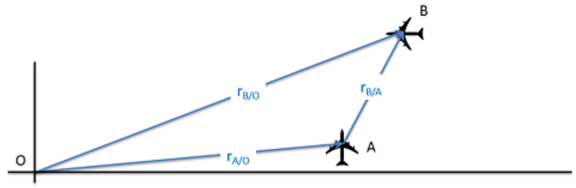

Figure 7.7.2: In two dimensions we have a fixed reference point O, as well as at least two moving bodies (labeled as A and B in this case). The $\vec{r}$ vectors represent the distance vectors between bodies, with the subscript indicating the starting and ending point. The subscripts should always be formatted end point / starting point.

In two dimensions, we will use the same equations as before, except this time we will be using vectors rather than scalar values in our equations.

$$\text{Position:} \quad \vec{r}_{B/O}=\vec{r}_{A/O}+\vec{r}_{B/A} \tag{7.7.4}$$

$$\text{Velocity:} \quad \vec{v}_{B/O}=\vec{v}_{A/O}+\vec{v}_{B/A} \tag{7.7.5}$$

$$\text{Acceleration:} \quad \vec{a}_{B/O}=\vec{a}_{A/O}+\vec{a}_{B/A} \tag{7.7.6}$$

To solve these equations, we will almost always break the vector equations down into a set of component equations. This will let us solve these equations with simple algebra. Because we have more than one particle, we usually use rectangular coordinate systems for this with a universal set of x and y coordinates.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/A5aJ2VYvwkw.

Example 7.7.1

A police officer notices a car speeding by. If the police car is traveling 30 m/s and the radar gun measures the relative velocity to be 15 m/s, how fast is the speeding car actually going? If the police car immediately begins accelerating at a constant rate and catches up to the speeding car after 15 seconds, what is the rate of acceleration of the police car?

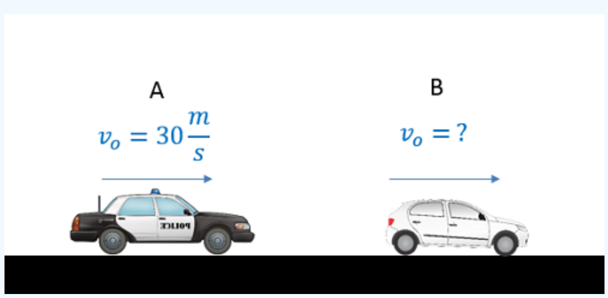

Figure 7.7.3: problem diagram for Example 7.7.1. A police car and a speeding car it is chasing after travel in the same direction on a straight road.

Solution

Video 7.7.2: Worked solution to example problem 7.7.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/YndovN5yWfc.

Example 7.7.2

A plane has an airspeed of 250 kilometers per hour (airspeed is the velocity of the plane relative to the air) and is flying though an easterly crosswind with a speed of 20 kilometers per hour. If the plane wants to maintain a direct northerly course, what is the angle the plane must point into the wind (θ)?

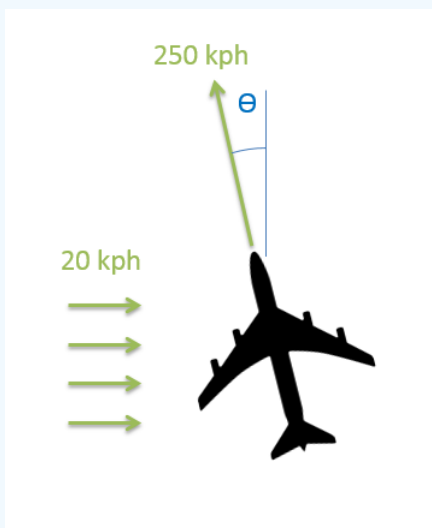

Figure 7.7.4: problem diagram for Example 7.7.2. A plane aiming to fly directly northwards experiences easterly crosswinds.

Solution

Video 7.7.3: Worked solution to example problem 7.7.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/ga1qcV38zYA.

? Example 7.7.3

You are in a boat that is traveling through the water in an area of swift currents. One instrument measures your speed with respect to the water to be 20 ft/s with your boat pointed at a 45-degree angle. GPS, however, measures your absolute speed and direction to be 25 ft/s at a 55-degree angle. Based on this information, what is the speed and direction of the water current in this area?

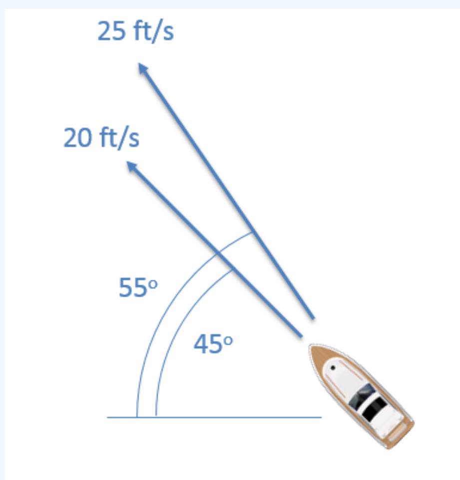

Figure 7.7.5: problem diagram for Example 7.7.3. Instruments on a boat return different readings for its absolute velocity vs. its velocity relative to the surrounding water.

Solution

Video 7.7.4: Worked solution to example problem 7.7.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/j8vnM_hrnQg.

7.8: Chapter 7 Homework Problems

Exercise 7.8.1

A car with an initial velocity of 30 m/s accelerates at a constant rate of 12 m/s² . Find the time required for the car to reach a speed of 60 m/s, and the distance traveled during this time.

Solution

Time = $2.5\ s$

Distance = $112.5\ m$

Exercise 7.8.2

A car traveling at 60 miles per hour approaches a fallen log in the road 400 feet away. Assuming the driver immediately slams on the brakes, what is the required rate of deceleration needed to assure the driver does not hit the log?

Solution

Minimum acceleration: $-9.68\ ft/s^2$

Exercise 7.8.3

A train experiences the acceleration over time detailed below. Draw the velocity-time and position-time diagrams with all key points and equations labeled, and determine the total distance traveled by the train.

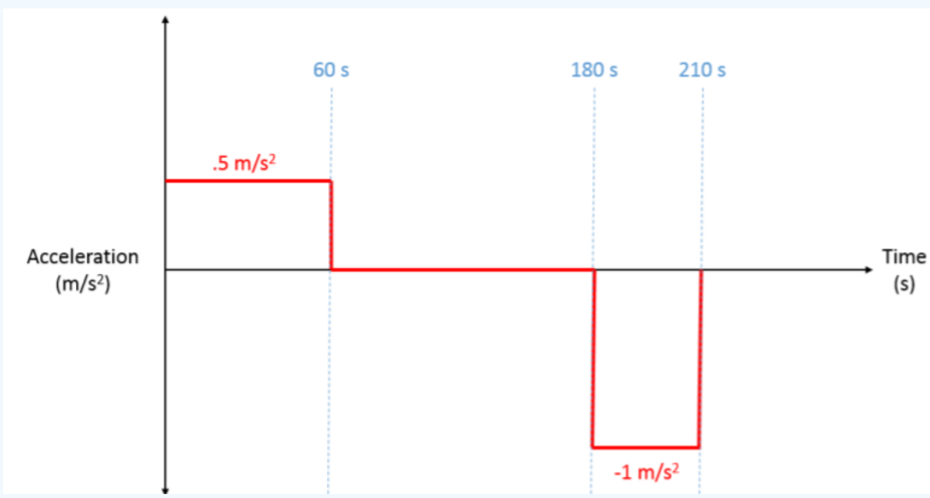

Figure 7.8.1: problem diagram for Exercise 7.8.3. Graph of a train's acceleration in m/s² over a 210-second period.

Solution

Total distance = $4950\ m$, plus v-t and s-t diagrams

Exercise 7.8.4

As a roller coaster cart comes into the gate at the end of the ride, it goes through two sets of brakes. The cart's velocity over time is shown in the graph below. Draw the acceleration-time and position-time diagrams with all key points and equations labeled. Determine the total distance the cart travels during this seven-second period.

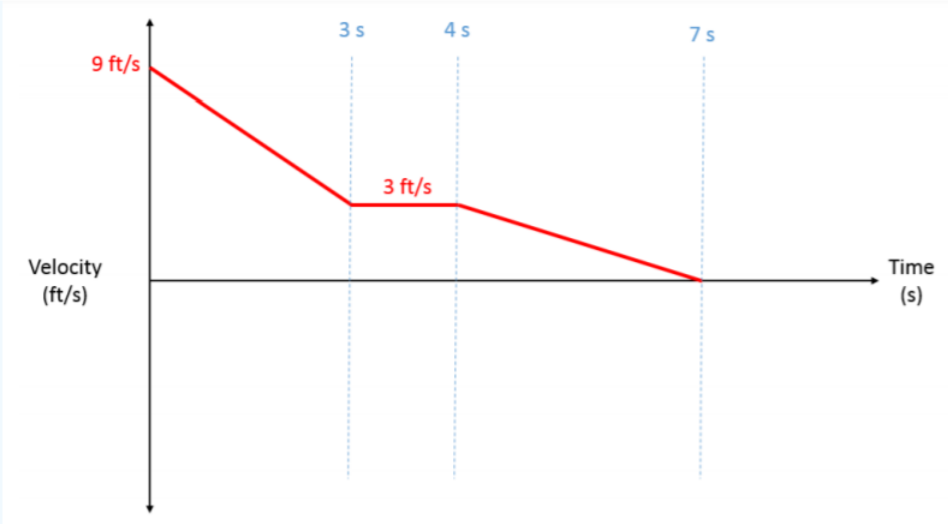

Figure 7.8.2: problem diagram for Exercise 7.8.4. Graph of a roller coaster cart's velocity in ft/s over a 7-second period.

Solution

Total distance = $25.5\ ft$; plus a-t and s-t diagrams.

? Exercise 7.8.5

A tank fires a round at a 30-degree angle with a muzzle velocity of 600 m/s. The round is expected to hit a mountainside one kilometer away. The mountainside also has an average angle of 30 degrees. How far up the mountainside will the round be expected to travel before hitting the ground (d) if we ignore air resistance?

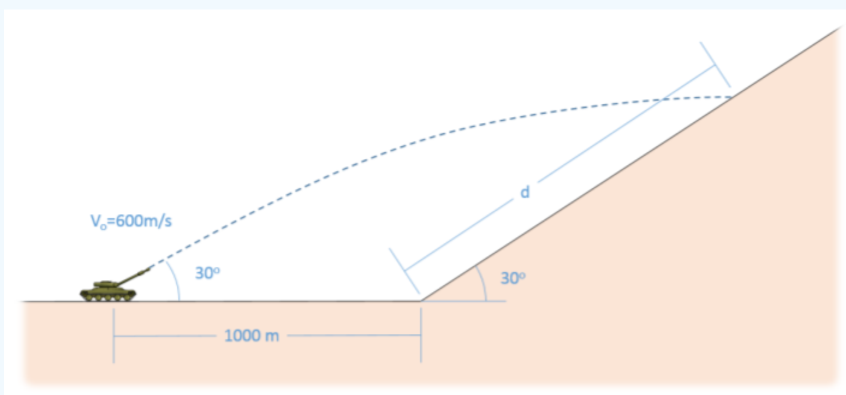

Figure 7.8.3: problem diagram for Exercise 7.8.5. A tank fires a round at an angle, aiming towards a mountainside 1 kilometer away.

Solution

$d = 5.37\ km$

? Exercise 7.8.6

A plane with a current speed of 600 ft/s is increasing in speed while also making a turn. The acceleration is measured at 40 ft/s² at an angle 35° from its current heading. Based on this information, determine the rate at which the plane is increasing its speed (tangential acceleration) and the radius of the turn for the plane.

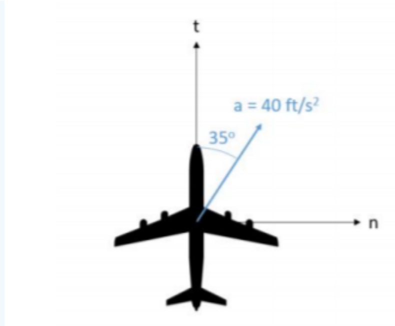

Figure 7.8.4: problem diagram for Exercise 7.8.6. A plane depicted in a normal-tangential coordinate system undergoes an acceleration that changes both its speed and its direction.

Solution

$a_t = 32.77 \; ft/s^2$

$r = 15,690 \; ft$

? Exercise 7.8.7

A radar station is tracking a rocket with a speed of 400 m/s and an acceleration of 3 m/s² in the direction shown below. The rocket is 3.6 km away (r = 3600 m) at an angle of 25°. What would you expect $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$ to be?

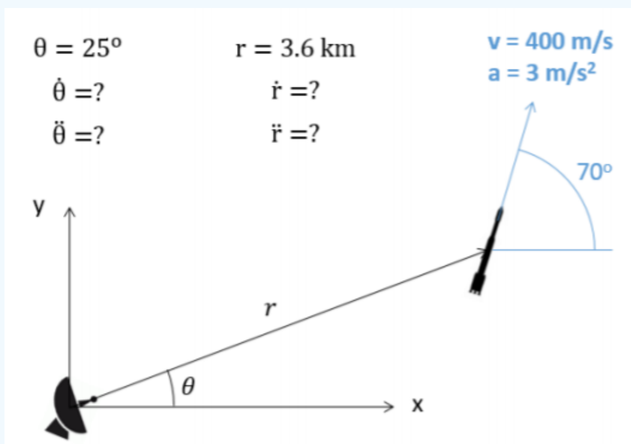

Figure 7.8.5: problem diagram for Exercise 7.8.7. A radar station tracks the instantaneous velocity and acceleration of a rocket located a known distance and angle away.

Solution

$\dot{r} = 282.8 \; m/s, \; \ddot{r} = 24.34 \; m/s^2$

$\dot{\theta} = 0.0786 \; rad/s, \; \ddot{\theta} \; -0.01176 \, rad/s^2$

? Exercise 7.8.8

The pulley system below is being used to lift a heavy load. Assuming the end of the cable is being pulled at a velocity of 2 ft/s, what is the expected upwards velocity of the load?

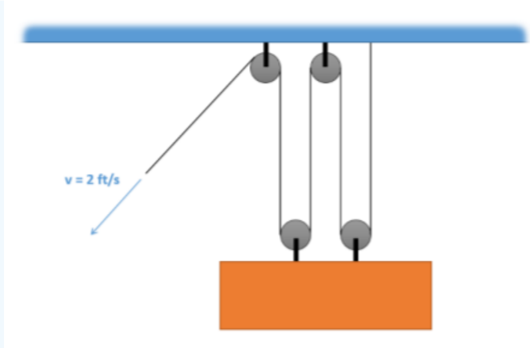

Figure 7.8.6: problem diagram for Exercise 7.8.8. A rope's right end is fastened to the ceiling, and it runs alternately through two pulleys mounted on a load and two pulleys mounted on the ceiling. The left end of the rope is pulled down to raise the load.

Solution

$v_L = 0.5\, ft/s$

Exercise 7.8.9

A cable is anchored at A, goes around a pulley on a movable collar at B, and finally goes around a pulley at C as shown below. If the end of the rope is pulled with a velocity of 0.5 m/s, what is the expected velocity of the collar at this instant?

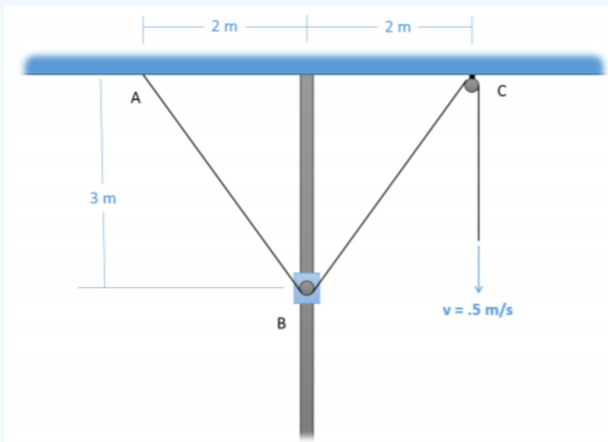

Figure 7.8.7: problem diagram for Exercise 7.8.9. A rope attached to the ceiling at one end runs through a pulley mounted on a sliding collar mounted on a post extending from the ceiling, and then through a pulley mounted on the ceiling. The free rope's free end is pulled downward, moving the collar.

Solution

$v_B = 0.3\, m/s$

Exercise 7.8.10

You are driving at a velocity of 90 ft/s in the rain while you notice that the rain is hitting your car at an angle 35° from the vertical, from your perspective. Assuming the rain is actually coming straight down (when observed by a stationary person), what is the velocity of the rain with respect to the ground?

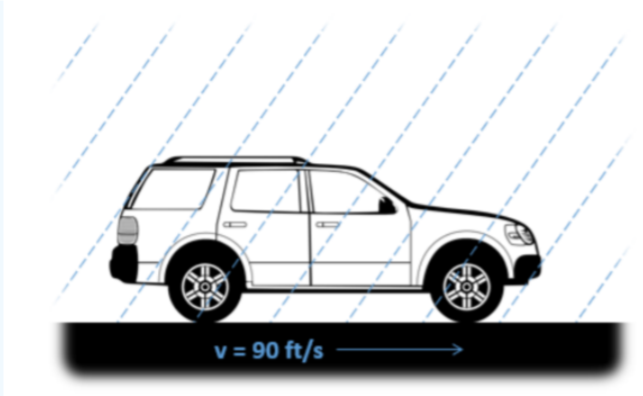

Figure 7.8.8: problem diagram for Exercise 7.8.10. Illustration of the rain's motion as seen from the perspective of a person in the car.

Solution

$v_r = 128.5\ ft/s$

CHAPTER OVERVIEW

8: Newton's Second Law for Particles

8.0: Video Introduction to Chapter 8

Video introduction to the topics covered in this chapter, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/HLy8efQ45qQ.

8.1: One-Dimensional Equations of Motion

Kinetics is the branch of dynamics that deals with the relationship between motion and the forces that cause that motion. The basis for all of kinetics is Newton's Second Law, which relates forces and accelerations for a given body. In its basic form, **Newton's Second Law** states that the sum of the forces on a body will be equal to the mass of that body times the rate of acceleration. For bodies in motion, we can write this relationship out as the equation of motion.

$$\sum \vec{F} = m * \vec{a} \tag{8.1.1}$$

In cases where accelerations only exist in a single dimension, we can reduce the above vector equation into a single scalar equation. Calling that single direction the x direction, we arrive at the single equation of motion shown below. By entering known forces or accelerations, we can use this equation to solve for a single unknown force or acceleration term.

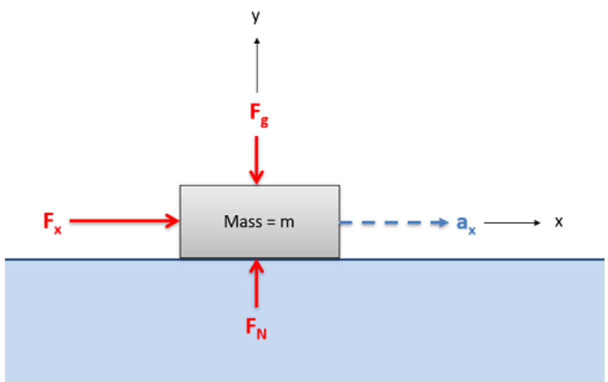

Figure 8.1.1: This box being pushed along a frictionless surface can be examined as a one dimensional kinetics problem. Acceleration exists only in the x direction, related by the equation of motion to the single unbalanced force in the x direction. Because the forces in the y direction are balanced, the acceleration in that direction will be zero.

$$\sum F_x = m * a_x = m * \ddot{x} \tag{8.1.2}$$

Kinetics and the **equation(s) of motion** relate forces and accelerations, and are often used in conjunction with the **kinematics** equations, which relate positions, velocities and accelerations as discussed in the previous chapter. Depending on the problem being examined, the kinematics equations may need to be examined either before or after the kinetics equations.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/CEL2cpvdQmQ.

? Example 8.1.1

A block with a weight of 90 pounds sits on a frictionless surface and a 50-pound force is applied in the x direction, as shown below.

- What is the rate of acceleration of the block?

- What is the velocity and displacement three seconds after the force is applied?

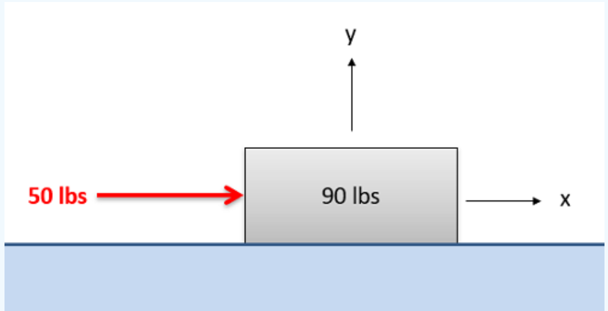

Figure 8.1.2: problem diagram for Example 8.1.1. A box on a flat, frictionless surface experiences a pushing force towards the right.

Solution

Video 8.1.2: Worked solution to example problem 8.1.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/NwmVFrTGd0A.

? Example 8.1.2

A block with a weight of 90 pounds sits on a surface with a kinetic coefficient of friction of 0.2, and a 50-pound force is applied in the x direction as shown below.

- What is the rate of acceleration of the block?
- What is the velocity and displacement three seconds after the force is applied?

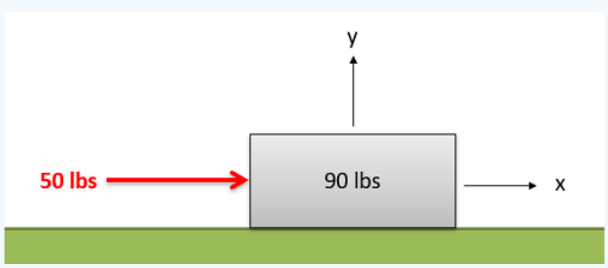

Figure 8.1.3: problem diagram for Example 8.1.2. A box on a flat surface, which produces friction against the box, experiences a pushing force towards the right.

Solution

Video 8.1.3: Worked solution to example problem 8.1.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/9BgdnzTUO9I.

Example 8.1.3

A 2000-lb elevator decelerates downward, going from a speed of 25 ft/s to a stop in a distance of 50 ft.

- What is the average rate of deceleration?
- What is the tension in the cable supporting the elevator during this period?

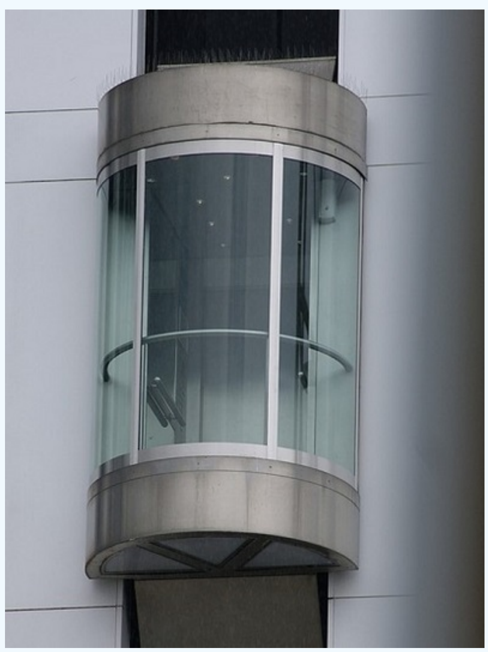

Figure 8.1.4: A descending glass-sided elevator.

Solution

Video 8.1.4: Worked solution to example problem 8.1.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/I2j0peOLzro.

Example 8.1.4

A rocket test sled is being used to test a solid rocket booster (mass = 1000 kg). It's known that generally a solid rocket booster's force will fit the equation $F = A + Bt - Ct^2$. If the rocket has an initial thrust of 10 kN, and achieves a speed of 150 m/s and travels 700 meters during a 10-second test run, determine the constants A, B and C for the rocket.

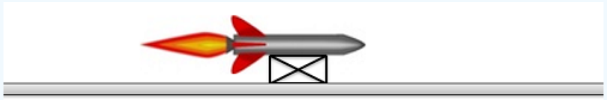

Figure 8.1.5: A rocket sled holding a solid booster, moving rightwards on a straight track.

Solution

Video 8.1.5: Worked solution to example problem 8.1.4, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/tmUOZXuAzNE.

8.2: Equations of Motion in Rectangular Coordinates

To start our discussion of kinetics in two dimensions, we will examine Newton's Second Law as applied to a fixed coordinate system. In its basic form, **Newton's Second Law** states that the sum of the forces on a body will be equal to the mass of that body times its rate of acceleration. For bodies in motion, we can write this relationship out as the equation of motion.

$$\sum \vec{F} = m * \vec{a} \tag{8.2.1}$$

With rectangular coordinates in two dimensions, we will break this single vector equation into two separate scalar equations. To solve the equations, we simply break any given forces and accelerations down into x and y components using sines and cosines and plug those known values in. With two equations, we should be able to solve for up to two unknown force or acceleration terms.

$$\sum F_x = m * a_x = m * \ddot{x} \tag{8.2.2}$$

$$\sum F_y = m * a_y = m * \ddot{y} \tag{8.2.3}$$

Just as with a single dimension, the equations of motion are often used in conjunction with the kinematics equations that relate positions, velocities and accelerations as discussed in the previous chapter. Depending on the problem being examined, the kinematics equations may need to be examined either before or after the kinetics equations.

Rectangular coordinates can be used in any kinetics problem; however, they work best with problems where the forces do not change direction over time. Projectile motion is a good example of this, because the gravity force will maintain a constant direction, as opposed to the thrust force on a turning plane, where the thrust force changes direction with the plane.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/2D15kYAYUYg.

Example 8.2.1

You are controlling a satellite with a mass of 300 kg. The main and lateral thrusters can exert the forces shown. How long do you need to run each of the thrusters to achieve the final velocity as shown in the diagram? Assume the satellite has zero initial velocity.

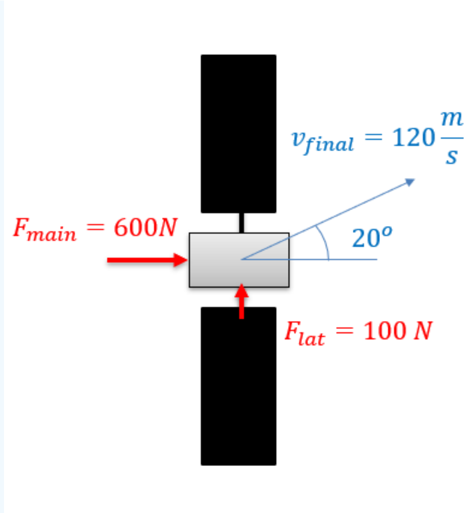

Figure 8.2.1: problem diagram for Example 8.2.1. A satellite facing the right experiences a rightwards force form its main thrusters and an upwards force from its lateral thrusters, changing both its speed and direction.

Solution

Video 8.2.2: Worked solution to example problem 8.2.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/YawjRTV89Lo.

? Example 8.2.2

A man in a flatbed truck that starts at rest moves up a hill at an angle of 10 degrees. If he is carrying a 600-kg crate in the back and the static coefficient of friction is 0.3, what is the maximum rate of acceleration before the crate slides off of the back of the truck? How long will it take the truck to reach a speed of 25 m/s?

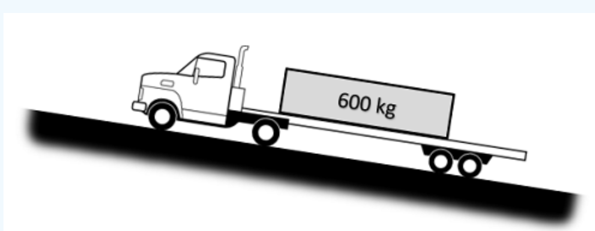

Figure 8.2.2: problem diagram for Example 8.2.2. A truck with a 600-kg load on its bed begins to move up a 10° incline.

Solution

Video 8.2.3: Worked solution to example problem 8.2.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/A4ZnX9vWfMA.

8.3: Equations of Motion in Normal-Tangential Coordinates

Continuing our discussion of kinetics in two dimensions, we can examine Newton's Second Law as applied to the normal-tangential coordinate system. In its basic form, **Newton's Second Law** states that the sum of the forces on a body will be equal to the mass of that body times the rate of acceleration. For bodies in motion, we can write this relationship out as the equation of motion.

$$\sum \vec{F} = m * \vec{a}$$

Just as we did with with rectangular coordinates, we will break this single vector equation into two separate scalar equations. This involves identifying the normal and tangential directions and then using sines and cosines to break the given forces and accelerations down into components in those directions.

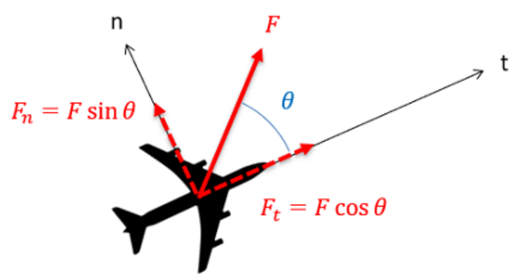

Figure 8.3.1: When working in the normal-tangential coordinate system, any given forces or accelerations can be broken down using sines and cosines as long as the angle of the force or acceleration is known relative to the normal and tangential directions.

$$\sum F_n = m * a_n \tag{8.3.1}$$

$$\sum F_t = m * a_t \tag{8.3.2}$$

Just as with rectangular coordinates, these equations of motion are often used in conjunction with the kinematics equations, which relate positions, velocities and accelerations as discussed in the previous chapter. In particular, we will often substitute the known values below for the normal and tangential components for acceleration.

$$a_n = v * \dot{\theta} = \frac{v^2}{\rho} \tag{8.3.3}$$

$$a_t = \dot{v} \tag{8.3.4}$$

Normal-tangential coordinates can be used in any kinetics problem; however, they work best with problems where forces maintain a consistent direction relative to some body in motion. Vehicles in motion are a good example of this: the direction of the forces applied are largely dependent on the current direction of the vehicle, and these forces will rotate with the vehicle as it turns.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/PK2swJu37sg.

? Example 8.3.1

A 1000-kg car travels over a hill at a constant speed of 100 kilometers per hour. The top of the hill can be approximated as a circle with a 90-meter radius.

- What is the normal force the road exerts on the car as it crests the hill?
- How fast would the car have to be going to get airborne?

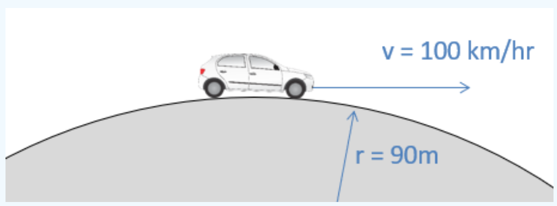

Figure 8.3.2: problem diagram for Example 8.3.1. A car is at the top of a hill, moving towards the right at a constant speed of 100 km/hr.

Solution

Video 8.3.2: Worked solution to example problem 8.3.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/KOLdXpQ5M1Q.

? Example 8.3.2

A 2500-pound car is traveling 40 feet per second. The coefficient of friction between the car's tires and the road is 0.9.

- If the car is maintaining a constant speed, what is the minimum radius of curvature before slipping?
- Assuming the car is speeding up at a rate of 10 ft/s², what is the minimum radius of curvature before slipping?

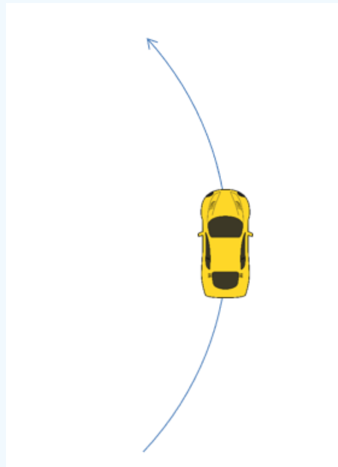

Figure $8.3.3$: problem diagram for Example $8.3.2$. A car travels around a circular path in a counterclockwise direction.

Solution

Video $8.3.3$: Worked solution to example problem $8.3.2$, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/Gw0_H0wdqm8.

? Example 8.3.3

15-kg boxes are being transported around a curve via a conveyor belt, as shown below. Assuming the curve has a radius of 3 meters and the boxes are traveling at a constant speed of 1 meter per second, what is the minimum coefficient of friction needed to ensure the boxes don't slip as they travel around the curve?

Figure 8.3.4: problem diagram for Example 8.3.3; a level section of conveyor belt in a semicircular shape.

Solution

Video 8.3.4: Worked solution to example problem 8.3.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/4z73Pc3s_TE.

8.4: Equations of Motion in Polar Coordinates

To finish our discussion of the equations of motion in two dimensions, we will examine Newton's Second law as it is applied to the polar coordinate system. In its basic form, **Newton's Second Law** states that the sum of the forces on a body will be equal to mass of that body times the rate of acceleration. For bodies in motion, we can write this relationship out as the equation of motion.

$$\sum \vec{F} = m * \vec{a}$$

Just as we did with with rectangular and normal-tangential coordinates, we will break this single vector equation into two separate scalar equations. This involves identifying the r and θ directions and then using sines and cosines to break the given forces and accelerations down into components in those directions.

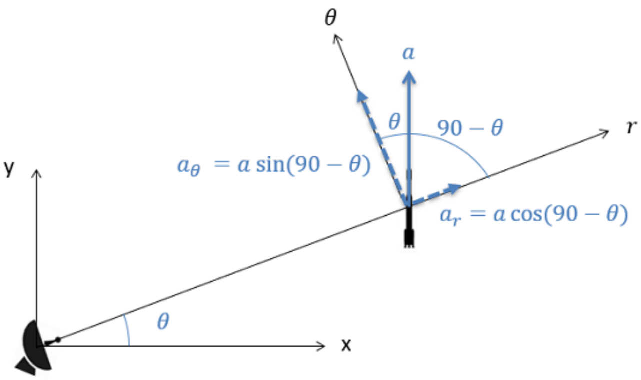

Figure 8.4.1: When working in the polar coordinate system, any given forces or accelerations can be broken down using sines and cosines as long as the angle of the force or acceleration is known relative to the r and θ directions.

$$\sum F_r = m * a_r \tag{8.4.1}$$

$$\sum F_\theta = m * a_\theta \tag{8.4.2}$$

Just as with our other coordinate systems, the equations of motion are often used in conjunction with the kinematics equations, which relate positions, velocities and accelerations as discussed in the previous chapter. In particular, we will often substitute the known values below for the r and θ components for acceleration.

$$a_r = \ddot{r} - r\theta^2 \tag{8.4.3}$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} \tag{8.4.4}$$

Polar coordinates can be used in any kinetics problem; however, they work best with problems where there is a stationary body tracking some moving body (such as a radar dish) or there is a particle rotating around some fixed point. These equations will also come back into play when we start examining rigid body kinematics.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/XiuQSSVdRKk.

Example 8.4.1

A device consists of two masses, each 0.5 kg in mass, tethered to a central shaft. The tethers are each 0.75 meters long and each tether currently makes a 25-degree angle with the central shaft. Assume the central shaft is spinning at a constant rate. What is the rate at which the shaft is spinning? If we want it to spin at exactly 100 rpm, what should the angle of the tethers be?

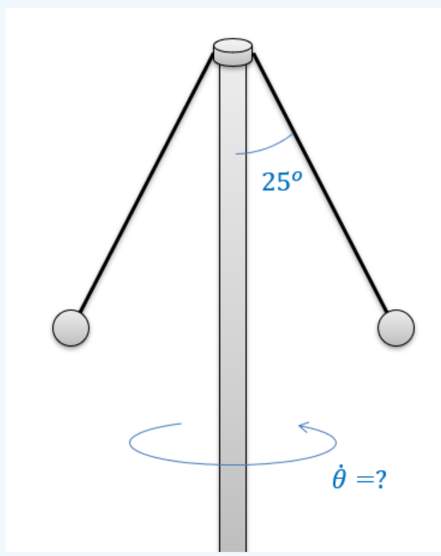

Figure 8.4.2: problem diagram for Example 8.4.1. A spinning shaft supports two identical tethers at its top end, with each tether holding a mass and splayed out symmetrically from the shaft.

Solution

Video 8.4.2: Worked solution to example problem 8.4.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/fGsoMdR1H9I.

? Example 8.4.2

A catapult design consists of a steel weight on a frictionless rod. The rod spins at a constant rate of 4 radians per second and when θ is 45 degrees from the horizontal, the 30-lb weight is released from its position 2 feet from the center of rotation of the shaft. What is the force the shaft exerts on the weight at the instant before and the instant after it is released? What is the acceleration of the weight along the shaft the instant after it is released?

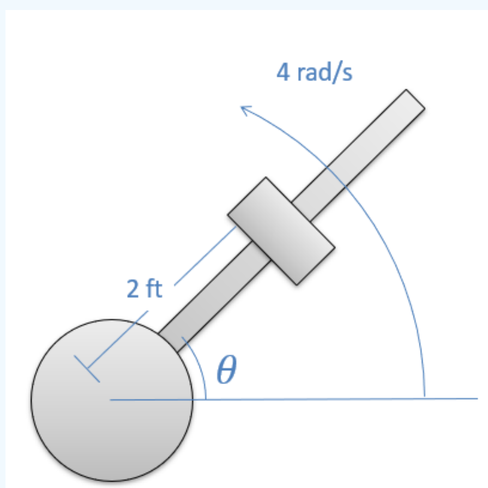

Figure 8.4.3: problem diagram for Example 8.4.2. A central vertical shaft that rotates supports a horizontal rod bearing a releasable weight.

Solution

Video 8.4.3: Worked solution to example problem 8.4.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/bKsi80wzLUY.

8.5: Chapter 8 Homework Problems

Exercise 8.5.1

A 50-kg box is being pulled across a surface by a force of 200 N in the direction shown below. If the coefficient of friction is 0.3, what is the rate of acceleration of the box and how far will the box move in a three-second period?

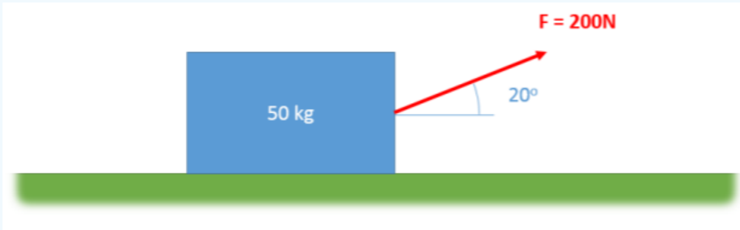

Figure 8.5.1: problem diagram for Exercise 8.5.1. A box is pulled across a horizontal surface by a force directed 20° above the horizontal.

Solution

$a = 1.23\,m/s^2$

$s = 5.52\,m$

Exercise 8.5.2

A 3-kg cannonball is shot out of a cannon with an initial velocity of 300 m/s at a 25-degree angle. A headwind exerts a constant 5 N horizontal force. How far will the cannonball travel horizontally before hitting the ground?

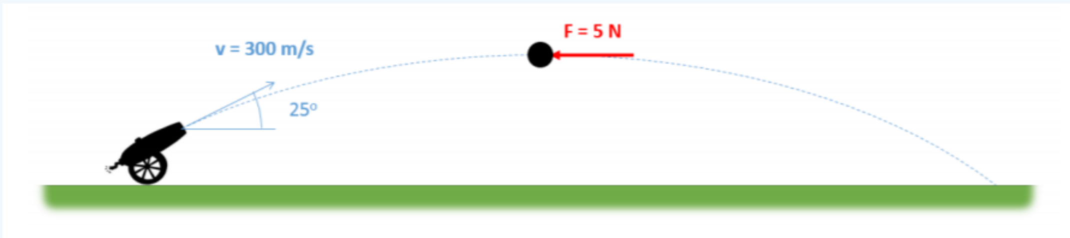

Figure 8.5.2: problem diagram for Exercise 8.5.2. Diagram of the parabolic path the cannonball is expected to travel before hitting the ground to the right of the cannon, with the leftwards headwind force on the ball shown.

Solution

$d = 6470\,m$

Exercise 8.5.3

A 1-kg block sits on a rotating table as shown below. If the static coefficient of friction is assumed to be 0.4, what is the maximum angular velocity $(\dot{\theta})$ that can be achieved before the block begins to slip?

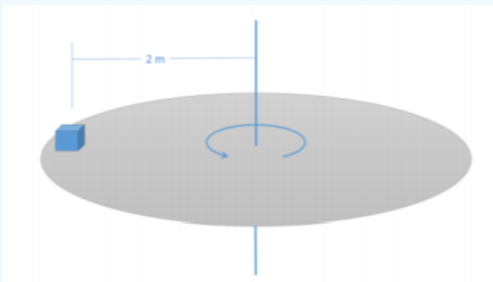

Figure 8.5.3: problem diagram for Exercise 8.5.3. A small block sits 2 meters from the center of a level, rotating circular table.

Solution

$\dot{\theta} = 1.4 \, \frac{rad}{s}$

❓ Exercise 8.5.4

A 5-kg instrument is held via a cable to a space station. The instrument and space station are both rotating at a rate of 0.5 rad/s when the space station begins retracting the cable at a constant rate of 0.25 m/s.

- What is the tension in the cable at this instant?
- What will the angular acceleration $(\ddot{\theta})$ of the cable be? Hint: there are no forces in the θ direction.

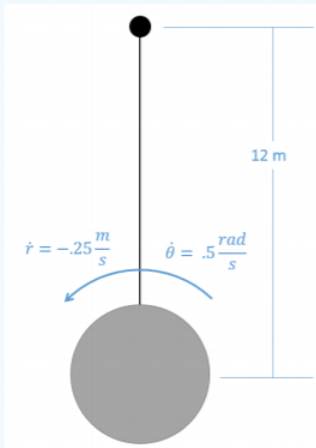

Figure 8.5.4: problem diagram for Exercise 8.5.4. A space station and a cable-connected instrument, rotating as a system, draw closer together as the station pulls in the cable at a constant rate.

Solution

$T = 15 \, N$

$\ddot{\theta} = 0.0208 \, \frac{rad}{s^2}$

CHAPTER OVERVIEW

9: Work and Energy in Particles

9.0: Video Introduction to Chapter 9

Video introduction to topics covered in this chapter, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/2uuVtzA_SFs.

9.1: Conservation of Energy for Particles

The concepts of **work** and **energy** provide the basis for solving a variety of kinetics problems. Generally, this method is called the **Energy Method** or the **Conservation of Energy**, and it can be boiled down to the idea that the work done to a body will be equal to the change in energy of that body. Dividing energy into kinetic and potential energy pieces as we often do in dynamics problems, we arrive at the following base equation for the conservation of energy.

$$W = \Delta KE + \Delta PE \tag{9.1.1}$$

It is important to notice that unlike Newton's Second Law, the above equation is **not** a vector equation. It does not need to be broken down into components which can simplify the process. However, we only have a single equation and therefore can only solve for a single unknown, which can limit the method.

Work:

To understand how to use the energy method we first need to understand the concepts of work and energy. **Work** in general is a force exerted over a distance. If we imagine a single, constant force pushing a body in a single direction over some distance, the work done by that force would be equal to the magnitude of that force times the distance the body traveled. If we have a force that is opposing the travel (such as friction), it would be negative work.

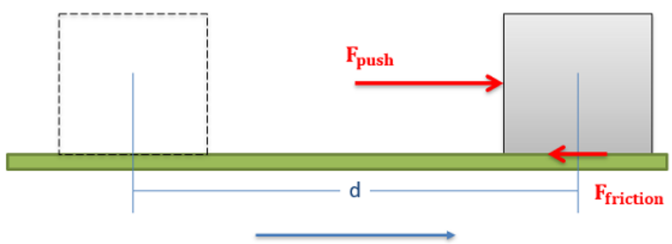

Figure 9.1.1: In instances with a constant force and a constant direction, the work done to a body will be equal to the magnitude of that force times the distance the body travels. For forces opposing the motion, the work will be negative.

$$W_{push} = F_{push} * d \tag{9.1.2}$$

$$W_{friction} = -F_{friction} * d \tag{9.1.3}$$

For instances where forces and the direction of travel do not match, the component of the force in the direction of travel is the only piece of the force that will do work. Following through with this logic, forces that are perpendicular to the direction of travel for a body will exert no work on a body because there is no component of the force in the direction of travel.

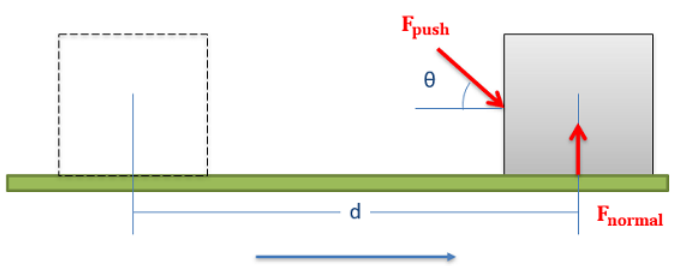

Figure 9.1.2: Only the components of a force in the direction of travel exert work on a body. Forces perpendicular to the direction of travel will exert no work on the body.

$$W_{push} = F_{push}\cos(\theta) * d \tag{9.1.4}$$

$$W_{normal} = 0 \tag{9.1.5}$$

In the case of a force that does not remain constant, we will need to account for the changing force. To do this we will simply integrate the force function over the distance traveled by the body. Just as before, only the component of the force in the direction of travel will count towards the work done, and forces opposing travel will be negative work.

$$W = \int_{x_1}^{x_2} F(x)\, dx \tag{9.1.6}$$

Energy:

When discussing energy in engineering dynamics, we will usually break energy down into **kinetic energy** and **potential energy**. Kinetic energy is the the energy mass in motion, while potential energy represents the energy that is stored up due to the position or stresses in a body.

In its equation form, the kinetic energy of a particle is represented by one half of the mass of the body times its velocity squared. If we wish to determine the change in kinetic energy, we would simply take the final kinetic energy minus the initial kinetic energy.

$$KE = \frac{1}{2} m v^2 \tag{9.1.7}$$

$$\Delta KE = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \tag{9.1.8}$$

As a note, a body that is rotating will also have rotational kinetic energy, but we will save that for our discussion of work and energy with rigid bodies.

Potential energy, unlike kinetic energy, is not really energy at all. Instead, it represents the work that a given force will potentially do between two instants in time. Potential energy can come in many forms, but the two we will discuss here are gravitational potential energy and elastic potential energy. These represent the work that the gravitational force and a spring force will do, respectively. We often use these potential energy terms in place of the work done by gravity or springs. When including these potential energy terms, it's important to not additionally include the work done by gravity or spring forces.

The change in gravitational potential energy for any system is represented by the product of the mass of the body, the value g (9.81 m/s^2 or 32.2 ft/s^2 on the earth's surface), and the vertical change in height between the start position and the end position. In equation form, this is as follows.

$$\Delta PE_{gravity} = m * g * \Delta h \tag{9.1.9}$$

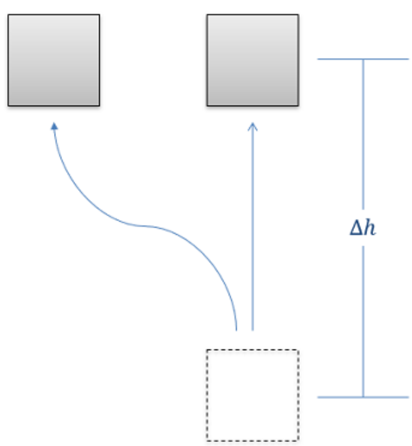

Figure 9.1.3: When finding the change in gravitational potential energy, we multiply the mass by g (giving us the weight of the object) and then multiply that by the change in the height of the object, regardless of the path taken.

To find the change in elastic potential energy, we will need to know the stiffness of the spring (represented by k, in units of force per distance) as well as the distance the spring has been stretched or compressed from its natural resting length (represented by x, in units of distance). Once we have those values, the elastic potential energy can be calculated by multiplying one half of the stiffness by the square of the distance x. To find the change in elastic potential energy, we simply take the final elastic potential energy and subtract the initial elastic potential energy.

$$\Delta PE_{spring} = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 \tag{9.1.10}$$

Going back to our original conservation of energy equation, we simply plug the appropriate terms on each side (work on the left and energies on the right) and balance the two sides to solve for any unknowns. Terms that do not exist or do not change, such as elastic potential energy in a problem with no springs or ΔKE in a problem where there is no change in the speed of the body, can be set to zero.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/FMkHrOjKlXs.

Example 9.1.1

A 16-pound crate slides down a ramp as shown below. The crate is released from a height of 10 feet above the ground.

- What is the work done by gravity?
- What is the change in gravitational potential energy?

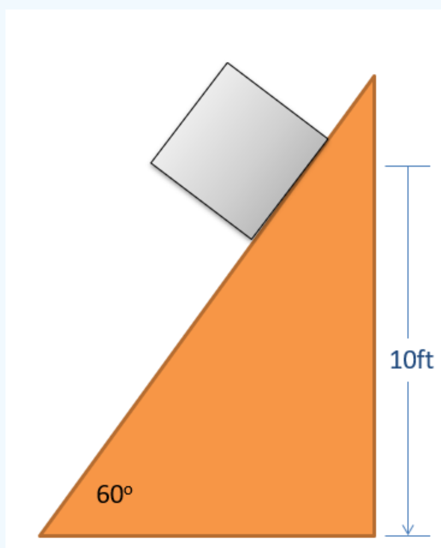

Figure 9.1.4: problem diagram for Example 9.1.1. A 16-lb crate initially 10 feet above the ground slides down a ramp of incline 60°.

Solution

Video 9.1.2: Worked solution to example problem 9.1.1. YouTube source: https://youtu.be/ZkopO1aTj54.

Example 9.1.2

A spring with an unstretched length of 40 cm and a k value of 120 N/cm is used to lift a 5-kg box from a height of 20 cm to a height of 30 cm. If the box starts at rest, what would you expect the final velocity to be?

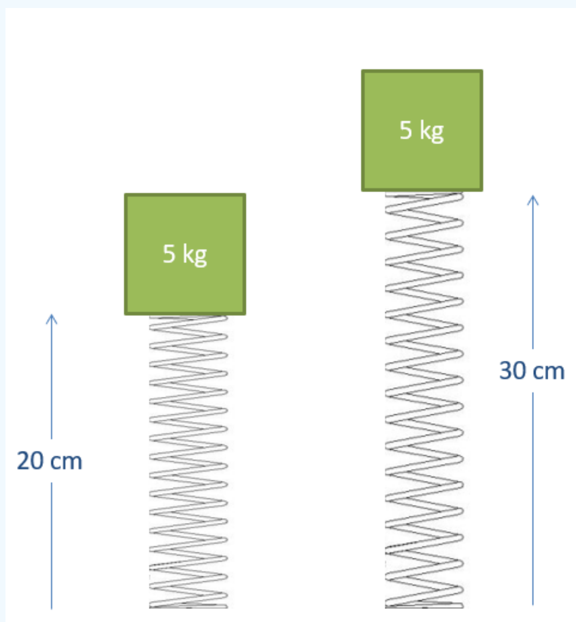

Figure 9.1.5: problem diagram for Example 9.1.2. A vertically oriented spring supporting a box on its upper end is stretched to lift the load.

Solution

Video 9.1.3: Worked solution to example problem 9.1.2. YouTube source: https://youtu.be/T_5JT4XFQN8.

Example 9.1.3

A 2,000-pound wrecking ball hangs from the end of a 40-foot cable. If the wrecking ball is released from an angle of 40 degrees from vertical, what would the expected maximum velocity at the bottom point of the travel path be?

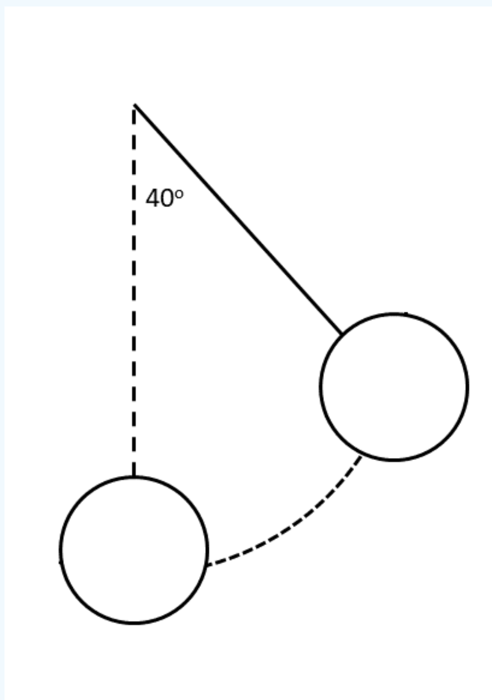

Figure 9.1.6: problem diagram for Example 9.1.3. A wrecking ball on a cable is raised slightly above its resting position, with the cable kept taut, then released.

Solution

Video 9.1.4: Worked solution to example problem 9.1.3. YouTube source: https://youtu.be/UkcpT1lfIDY.

Example 9.1.4

A 24,000-kilogram aircraft is launched from an aircraft carrier using a hydraulic catapult. If the force the catapult exerts over the 90-meter runway is shown in the graph below:

- What is the work done by the catapult?
- What is the speed of the plane at the end of the runway?

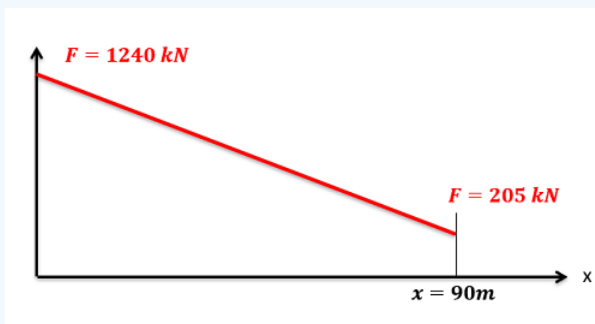

Figure 9.1.7: problem diagram for Example 9.1.4. Graph of the force exerted by the catapult vs distance traveled over the runway, showing a linear relationship between these quantities.

Solution

Video 9.1.5: Worked solution to example problem 9.1.4. YouTube source: https://youtu.be/DfSMeUm1SG8.

edit history is available upon request.

9.2: Power and Efficiency for Particles

Related to the concepts of work and energy are the concepts of **power** and **efficiency**. At its core, power is the rate at which work is being done, and efficiency is the percentage of useful work or power that is transferred from the input to the output of some system.

Power

Power at any instant is defined as the derivative of work with respect to time. If we look at the average power over a set period, we can simply measure the work done and divide that by the time. Work is defined as the force times the distance traveled, and distance over time is the velocity of a object, giving us many possible options for relating power, work, force, distance, time, and velocity.

$$P = \frac{dW}{dt} \tag{9.2.1}$$

$$P_{ave} = \frac{W}{t} = \frac{F * d}{t} = F * v \tag{9.2.2}$$

The common units of power are **watts** for the metric system, where one watt is defined as one joule per second, or one Newton-meter per second, and **horsepower** in the English system, where one horsepower is defined as 550 foot-pounds per second. Maximum power ratings are often a primary specification for motors and engines, as gear trains can easily change the torque provided by a motor but the overall power will not be altered by gearing.

Figure 9.2.1: Assuming the two cars above have the same mass, it would take the same amount of work to get them up to a set speed (such as 60 miles per hour). However, the more powerful car would be able to get to this speed in a much shorter time period.

Efficiency

Any devices with work/power inputs and outputs will have some loss of work or power between that input and output, due to things like friction. While energy is always conserved, some energies such as heat may not be considered useful. A measure of the useful work or power that makes it from the input of a device to the output is the efficiency. Specifically, efficiency is defined as the work out of a device divided by the work put into the device. With power being the work over time, efficiency can also be described as power out divided by the power in to a device (the time term would cancel out, leaving us with our original definition).

$$\eta = \frac{W_{out}}{W_{in}} = \frac{P_{out}}{P_{in}} \tag{9.2.3}$$

It is impossible to have efficiencies greater than one (or 100%) because that would be a violation of the conservation of energy; however, for most devices we wish to get the efficiencies as close to one as possible. This is not only because greater efficiencies waste less work/power, but also because any work or power that is "lost" in the device will be turned into heat that may build up.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/BxK5D1DnD1Q.

Example 9.2.1

If a car delivers an average 100 horsepower to the road and weighs a total of 1.2 tons, how long will it take to go from 0 to 60 mph?

Figure 9.2.2: A car driving quickly down a straight, tree-lined road.

Solution

Video 9.2.2: Worked solution to example problem 9.2.1. YouTube source: https://youtu.be/ju7TOLHWEZg.

Example 9.2.2

Your car broke down and now needs to be repaired. How much power is required for a lift to raise your 1.2 ton car 6 feet off the ground in 15 seconds?

Figure 9.2.3: A car in a mechanic's shop is raised off the ground by a lift.

Solution

Video 9.2.3: Worked solution to example problem 9.2.2. YouTube source: https://youtu.be/BCVOzDwPR7c.

Example 9.2.3

The drag force of air on a car is equal to

$$F_d = \frac{1}{2}\rho v^2 c_d A$$

where ρ is the is the density of the air, v is the velocity, c_d is the drag coefficient, and A is the frontal area. If a Mazda RX7 has a drag coefficient of 0.29, a frontal area of 5.95 square feet, and a max power output of 146 hp, and the density of air is 0.002326 slug/ft³, what is the theoretical top speed of the Mazda assuming it only has to fight wind resistance?

Figure 9.2.4: A red Mazda RX7.

Solution

Video 9.2.4: Worked solution to example problem 9.2.3. YouTube source: https://youtu.be/DaJIXMjRfZo.

9.3: Conservation of Energy for Systems of Particles

Just as we used the energy method for a single particle, we can also use the energy method for a system of particles. As a reminder, the conservation of energy equation states that the change in energy of a body (including kinetic and potential energies) will be equal to the work done to a body between during that time.

$$W = \Delta KE_{\Delta} PE \tag{9.3.1}$$

For a system of particles, the sum of the work done to all particles will be equal to the change in energy of all particles collectively, essentially combining multiple conservation of energy equations into one.

$$\sum W = \sum \Delta KE + \sum \Delta PE \tag{9.3.2}$$

This would seem to make one complex equation out of multiple simple conservation of energy equations (applying the conservation of energy separately to each body), but there is an advantage in that **internal forces** in the system will cancel out. In the diagram below, we can see a system of two particles connected via a cable. Examining the bodies separately, we would have two tension forces and a friction force all doing work to one box or the other. In the single equation for the system of boxes, however, the work done by the two tension forces (one positive and one negative) will sum up to zero. This will be true for any forces that are exerted between the bodies in the system, and forces like these are known as internal forces. The friction force, on the other hand, is an example of an external force, in that it exists between the top box and the surface (which is not part of our system).

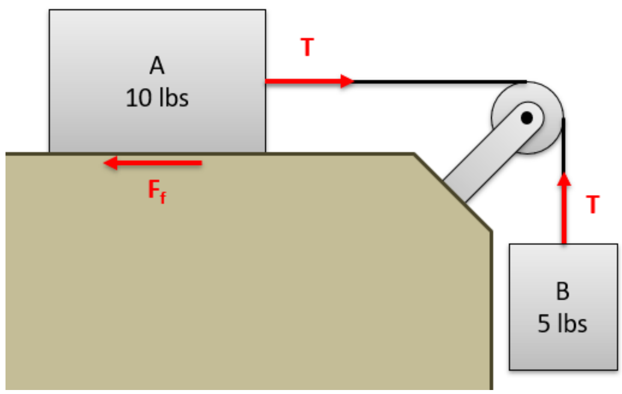

Figure 9.3.1: If we count the two boxes in this diagram as our system, then the tension forces would count as internal forces and can be ignored in our energy conservation equation. Only external forces such as the friction force need to be included as part of the work done on the system.

In the end, the sum of the work done by **external forces** will be equal to the change in total energy for the system of particles. Since we only have a single equation, we can only solve for a single unknown. We will often have to go back to our kinematics equations to relate the velocities and displacements of the various bodies to one another. Since these systems often consist of bodies connected to one another via cables, dependent motion analysis in particular will often come into play.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/r1RBsxURcbY.

Example 9.3.1

Two blocks are connected by a massless rope and a frictionless pulley as shown below. If the coefficient of friction between block A and the surface is 0.4, what is the speed of the blocks after block A has moved 6 ft?

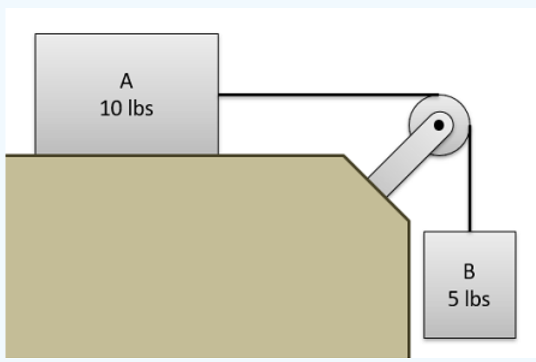

Figure 9.3.2: problem diagram for Example 9.3.1. The 5-lb box B hangs from a cable that runs over a pulley and connects to the 10-lb box A sitting on a flat surface.

Solution

Video 9.3.2: Worked solution to example problem 9.3.1. YouTube source: https://youtu.be/UW4HXJNId0A.

Example 9.3.2

The elevator shown below has a mass of 1500 kg and the counterweight has a mass of 500 kg. At some point the cable attached to the motor snaps, causing the elevator to begin falling. After falling 3 meters with no outside forces, what is the speed of the elevator? If the emergency brake is then applied at this point (3 m below the original position), exerting a constant force of 15,000 N, how much farther will the elevator fall before coming to a stop?

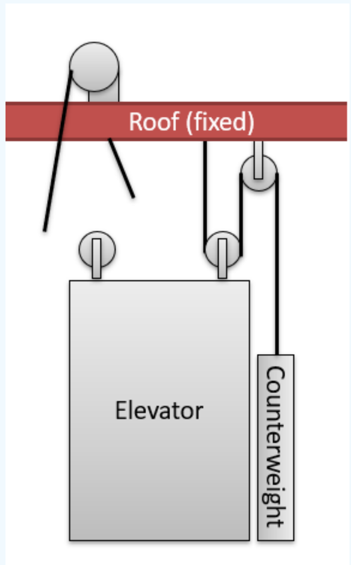

Figure 9.3.3: problem diagram for Example 9.3.2. An elevator is attached to the roof of a building only by a single cable, which is connected by pulleys to a counterweight at its free end.

Solution

Video 9.3.3: Worked solution to example problem 9.3.2. YouTube source: https://youtu.be/QWvh5wZt7jM.

9.4: Chapter 9 Homework Problems

Exercise 9.4.1

A car with a mass of 1100 kg locks up its brakes when it is traveling at 50 km/hr, stopping over a distance of 18 meters. If the same car were to lock up its brakes when traveling 80 km/hr, how far would you expect the car to slide before coming to a stop? (Hint: assume the same friction force in both cases).

Figure 9.4.1: A car with locked-up brakes, visibly skidding.

Solution

$d = 46.06\,m$

Exercise 9.4.2

A 2500-lb car traveling 60 mph (88 ft/s) impacts a highway crash barrier as shown below. If the barrier were designed to exert the following force over the 40-ft distance of the barrier, how far would you expect the car to travel after impacting the barrier?

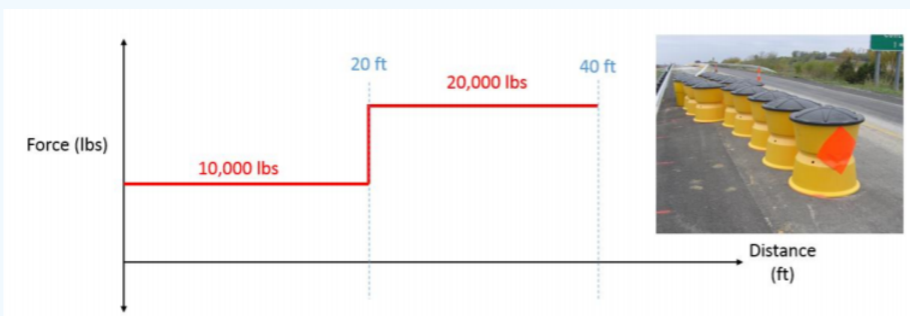

Figure 9.4.2: problem diagram for Exercise 9.4.2. A highway crash barrier and the graph of the force it exerts on an impacting car over the barrier's length.

Solution

$d = 25.03\ ft$(assuming no holes in the barrier)

Exercise 9.4.3

The Duquesne Incline transports passengers up a 30.5 degree slope. If a fully loaded car has a mass of 5500 kg, what power is required to maintain an uphill speed of 10 km/hr?

Figure 9.4.3: The Duquesne Incline, a cable car that transports passengers up a steep hillside.

Solution

$P = 76.13\,kW$

Exercise 9.4.4

A bungee jumper with a weight of 150 lbs uses a bungee cord with an unstretched length of 60 feet.

- Assuming no air resistance, what will the jumper's velocity be just before the bungee cord starts to stretch?
- If the bungee jumper falls a maximum distance of 150 feet, what is the spring constant of the bungee cord?

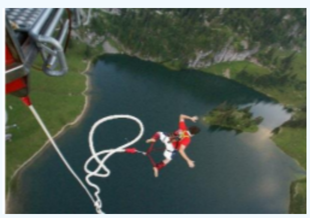

Figure 9.4.4: A man bungee-jumping above a lake.

Solution

$v = 62.16\,ft$

$k = 5.55\,lb/ft$

Exercise 9.4.5

An 1100-kg truck is being used to raise a 100-kg box using the setup shown below. When the box is at a height of 3 meters, the box has a velocity of 1 m/s.

- How far did the truck travel to lift the box this high? (Hint: this is a dependent motion problem)
- What is the velocity of the truck at this time?
- What is the work that the truck has done over this time?

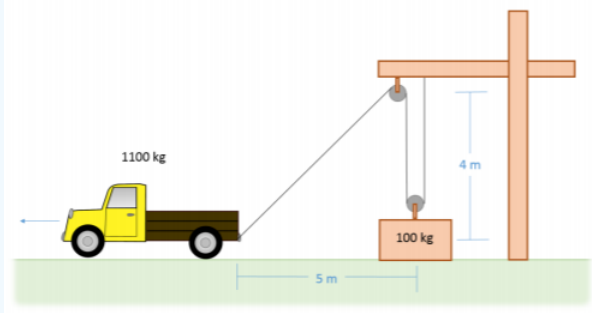

Figure 9.4.5: problem diagram for Exercise 9.4.5. A cable attached at one end to a beam 4 feet above the ground passes through a pulley on a box on the ground and through another pulley on the beam, before being attached to the rear of a truck 5 feet away from the box. The truck drives away from the box to raise it from the ground.

Solution

$d = 6.7\,m$

$v = 2.12\,m/s$

$W = 5464.92\,J$

CHAPTER OVERVIEW

10: Impulse and Momentum in Particles

10.0: Video Introduction to Chapter 10

Video introduction to the topics covered in Chapter 10, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/Xq39c7jZWPo.

10.1: Impulse-Momentum Equations for a Particle

The concepts of **impulse** and **momentum** provide a third method of solving kinetics problems in dynamics. Generally this method is called the **Impulse-Momentum Method**, and it can be boiled down to the idea that the impulse exerted on a body over a given time will be equal to the change in that body's momentum. The impulse is usually denoted by the variable J (not to be confused with the polar moment of inertia, which is also J) and the momentum is a body's mass times its velocity. Impulses and velocities are both vector quantities, giving us the basic equation below.

$$\vec{J}=m\vec{v}_f-m\vec{v}_i \tag{10.1.1}$$

For two-dimensional problems, we can break the single vector equation down into two scalar components to solve. In this case, we simply need to break all forces and velocities into x and y components.

$$J_x=mv_{f_x}-mv_{i_x} \tag{10.1.2}$$

$$J_y=mv_{f_y}-mv_{i_y} \tag{10.1.3}$$

Impulse:

The concept of an **impulse** in its most basic form is a force integrated over a time. For a force with a constant magnitude, we can find the magnitude of the impulse by multiplying the magnitude of the force by the time that force is exerted. If the force is not constant, we simply integrate the force function over the set time period. The direction of the impulse vector will be the direction of the force vector and the units will be a force times a time (Newton-seconds or pound-seconds, for example).

$$\text{Constant Magnitude Force:}\quad \vec{J}=\vec{F}*t \tag{10.1.4}$$

$$\text{Non-Constant Magnitude Force:}\quad \vec{J}=\int\vec{F}(t)\,dt \tag{10.1.5}$$

In many cases, we will discuss **impulsive forces**. This is an instance where we have very large forces acting over a very short time frame. In instances of impulsive forces, it is often difficult to measure the exact magnitude of the force or the time. In these cases we may only be able to deduce the magnitude of the impulse as a whole via the observed change in momentum of the body.

Figure 10.1.1: The force the tennis racket exerts on the ball will be very large, but it will be exerted over a very short period of time. Because of this, the force is considered an "impulsive" force. It would be difficult to determine the exact magnitude of the force or time frame of the impact, but by examining the velocity of the ball before and after the impact we could deduce the overall magnitude of the impulse as a whole. Photo by David Iliff. License: CC BY-SA 3.0.

Momentum:

The momentum of a body will be equal to the mass of the body times its current velocity. Since velocity is a vector, the momentum will also be a vector, having both magnitude and a direction. Unlike the impulse, which happens over some set time, the momentum is captured as a snapshot of a specific instant in time (usually right before and after some impulse is exerted). The units for momentum will be mass times unit distance per unit time. This is usually kilogram-meters per second in metric, or slug feet per second in English units.

Conservation of Momentum:

In instances where there is no impulse exerted on a body, we can use the original equation to deduce that there will be no change in momentum of the body. In this instance, momentum is conserved. This will also hold for systems of bodies, where if no **external** impulses are exerted on the bodies in a system, the momentum will be conserved as a whole. This is the basis of analysis for many collisions, as is discussed in the following sections.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/yC9wt53ho9k.

Example 10.1.1

A tennis ball (0.06 kg) is served to a tennis player at a speed of 10 m/s. The player then returns the ball at a speed of 36 m/s.

- What is the impulse exerted on the ball?
- If a high-speed camera reveals the impact lasted 0.02 seconds, what is the average force exerted on the ball during the collision?

Figure 10.1.2: A player in a tennis match serves a ball.

Solution

Video 10.1.2: Worked solution to example problem 10.1.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/WzD-ZyJy-T4.

Example 10.1.2

A plane with a mass of 80,000 kg is traveling at a velocity of 200 meters per second when the engines cut out. Twenty seconds later, it's noticed that the velocity has dropped to 190 m/s. Assuming the plane is not gaining or losing altitude, what is the average drag force on the plane?

Figure 10.1.3: An airliner in flight.

Solution

Video 10.1.3: Worked solution to example problem 10.1.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/dzhIT3r3u3k.

Example 10.1.3

The plot below shows the thrust generated by the engine on a jet fighter (mass of 2500 kg) over ten seconds. If the plane is starting from rest on a runway, and friction and drag are negligible, determine the speed of the plane at the end of these ten seconds.

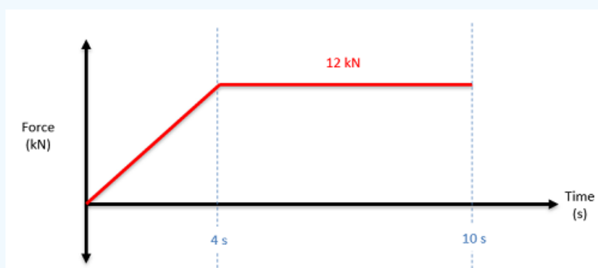

Figure 10.1.3: problem diagram for Example 10.1.3. Graph of the thrust force generated by a jet engine for the first 10 seconds of its motion starting from rest.

Solution

Video 10.1.4: Worked solution to example problem 10.1.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/XGxVjLa6wqQ.

10.2: Surface Collisions and the Coefficient of Restitution

Here we will use the term **surface collision** to describe any instance where a body impacts and rebounds off a solid and unmoving surface. A clear example of a surface collision is a basketball bouncing off a hard floor. The basketball will have some velocity before the collision and some second velocity after the collision, with the floor exerting an impulsive force during the collision that causes this change in velocity.

To analyze this collision, we will first need to set up a normal direction (perpendicular to the surface) and a tangential direction (parallel to the surface) for our problem, and break our velocities down into components in these directions. Assuming minimal friction during the impact, we will have an impulsive impact force acting entirely in the normal direction. This fact will form the basis for our analysis.

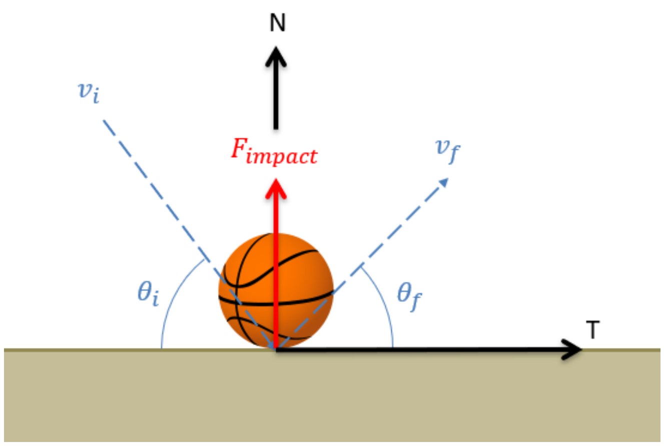

Figure 10.2.1: In a surface collision, the impulsive collision force will act in the normal direction. Because there is no force in the tangential direction, the velocity in the tangential direction will not change.

Because the impact force acts entirely in the normal direction, there will be no other significant force to change the momentum of the body in the tangential direction. Assuming the mass remains constant for the body, this means that the **velocity must remain constant in the tangential direction** because of the conservation of momentum.

$$v_{i,t} = v_{f,t} \tag{10.2.1}$$

To relate the velocities in the normal direction before and after the collision, we will use something called the **coefficient of restitution**. The coefficient of restitution is a number between 0 and 1 that measures the "bounciness" of the body and the surface in the collision. Specifically, for a single body being bounced perpendicularly off a surface, the coefficient of restitution is defined as the speed of the body immediately after bouncing off the surface divided by the speed immediately before bouncing off the surface. If we use velocities in place of speed, we will put a negative sign in our equation because the bounce causes a change in direction for the body.

$$\epsilon = -\frac{v_f}{v_i} \tag{10.2.2}$$

In instances where the body is being bounced off the surface at an angle, the impact force is entirely in the normal direction and the coefficient of restitution relationship specifically applies to the components of the velocities in the normal direction.

$$\epsilon = -\frac{v_{f,n}}{v_{i,n}} \tag{10.2.3}$$

This relationship can be applied to elastic collisions (where ϵ would be equal to 1), semi-elastic collisions (where ϵ would be some number between 0 and 1) and inelastic collisions (where ϵ would be equal to 0).

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/MaY6YEWhwJc.

Example 10.2.1

A basketball with an initial speed of 3 meters per second impacts a hard floor at the sixty degree angle as shown below. If the collision has a coefficient of restitution of 0.8, what is the expected speed and angle of the basketball after the impact?

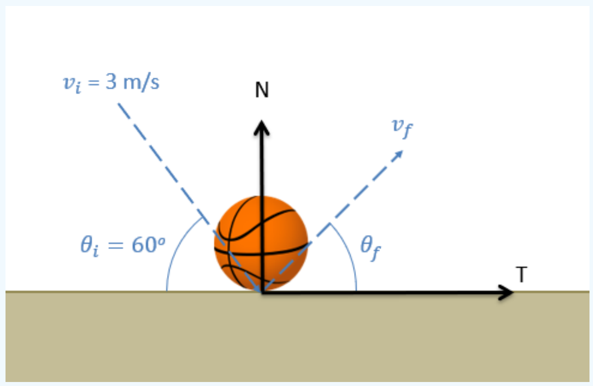

Figure 10.2.2: problem diagram for Example 10.2.1. A basketball with a known initial velocity bounces off a floor with a known coefficient of restitution.

Solution

Video 10.2.2: Worked solution to example problem 10.2.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/A8hl3JPUkB4.

Example 10.2.2

A bounce test is used to sort ripe cranberries from unripe cranberries. In this test, cranberries are dropped vertically onto a steel plate sitting at a 45-degree angle. After the impact, a cranberry is observed to bounce off at an angle of 20 degrees below the horizontal. Based on this information, what is the coefficient of restitution for the cranberry?

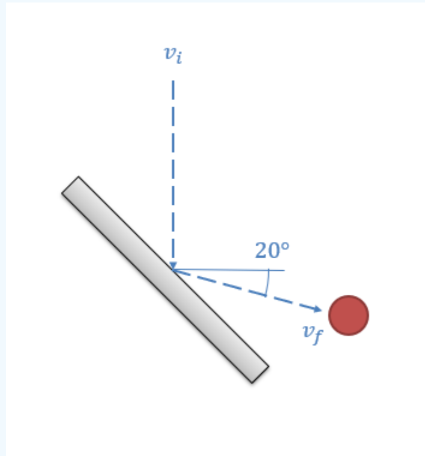

Figure 10.2.3: problem diagram for Example 10.2.2. A cranberry falls straight down onto a tilted plate and bounces off at a known angle.

Solution

Video 10.2.3: Worked solution to example problem 10.2.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/wpPYTx52CD8.

10.3: One-Dimensional Particle Collisions

The Impulse-Momentum Method is particularly useful when examining collisions between bodies. When examining the bodies colliding as a **system of particles**, such as the two bodies colliding below, the impulses exerted by **internal forces**, or the forces exerted between two bodies within the system, will be equal and opposite and hence cancel out in our impulse-momentum equation.

Figure 10.3.1: When two bodies collide, Newton's Third Law ensures that the normal forces from the collision will always be equal and opposite. This means that the impulses will be equal and opposite and will cancel out in our impulse-momentum equation when examining the system of equations.

Because the **impulsive** forces of the collision are so large over the very short period of the collision, other external forces are typically regarded as insignificant. This means that for our collision, there is no external impulse and therefore there is no change to the sum of the momentums of the bodies. This is known as the **conservation of momentum**, and it will hold true for all types of collisions.

$$\sum \vec{J} = 0 = \sum m\vec{v}_f - \sum m\vec{v}_i \tag{10.3.1}$$

$$\sum m\vec{v}_f - \sum m\vec{v}_i \tag{10.3.2}$$

In the case of the collision of two bodies, body A and body B in this case, we can break apart the sums as follows. The subscripts in this case are being used to denote both the body (A or B) and the initial versus final states (before or after the collision).

$$m_A\,\vec{v}_{A,f} + m_B\,\vec{v}_{B,f} = m_A\,\vec{v}_{A,i} + m_B\,\vec{v}_{B,i} \tag{10.3.3}$$

In addition to the conservation of momentum equation, we will also usually generate a second equation we can use alongside the conservation of momentum. This second equation will depend on the type of collision, though, with the three possible collision types being **elastic**, **inelastic**, and **semi-elastic**.

Elastic Collisions

An elastic collision is a collision in which all energy is assumed to be conserved as kinetic energy. In reality, true elastic collisions do not exist as some energy will always be converted to heat or sound, but in practice two very rigid bodies that collide without much deformation can get very close to the ideal of an elastic collision. An example of a collision that is close to elastic is a set of billiard balls colliding, as in Figure $10.3.1$.

An elastic collision conserves energy in addition to momentum, so the conservation of energy equation will be our second equation that we use to supplement the conservation of momentum.

$$\frac{1}{2}m_A\,\vec{v}_{A,f}^{2} + \frac{1}{2}m_B\,\vec{v}_{B,f}^{2} = \frac{1}{2}m_A\,\vec{v}_{A,i}^{2} + \frac{1}{2}m_B\,\vec{v}_{B,i}^{2} \tag{10.3.4}$$

Inelastic Collisions

An inelastic collision is the opposite of an elastic collision, in that much of the energy of the system is lost in the deformation of the bodies. In fact, in an inelastic collision the bodies must deform in such as way that they stick together after the collision. With the two bodies stuck together, they will have matching final velocities.

$$\vec{v}_{A,f} = \vec{v}_{B,f} \tag{10.3.5}$$

Semi-Elastic Collision

Finally, anything between an elastic collision and an inelastic collision is considered a semi-elastic collision. This is a collision where less than one hundred percent of the kinetic energy is conserved, but the objects do not stick together following the collision. In cases of semi-elastic collisions, we will use something called the **coefficient of restitution** (usually denoted by the Greek letter epsilon, ϵ), to supplement our conservation of momentum equation.

The coefficient of restitution is a number between 0 and 1 that measures the "bounciness" of the two bodies in the collision. Specifically, for a single body bouncing off a rigid surface, the coefficient of restitution is defined as the negative of the velocity after bouncing off the surface divided by the velocity before bouncing off the surface.

$$\epsilon = -\frac{v_f}{v_i} \tag{10.3.6}$$

A perfectly elastic collision would have a coefficient of restitution of one (no velocity would be lost), while a totally inelastic collision would have a coefficient of restitution of zero. All other collisions will have a coefficient of restitution that lies somewhere in between.

For cases where two bodies are bouncing off of one another, we would simply use relative velocities rather than the velocity of a single body. Using the velocity of body A relative to the velocity of body B, we come up with the equation below.

$$\epsilon = -\frac{\vec{v}_{f,A/B}}{\vec{v}_{i,A/B}} = -\frac{\vec{v}_{f,A} - \vec{v}_{f,B}}{\vec{v}_{i,A} - \vec{v}_{i,B}} \tag{10.3.7}$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/lFZ8Mr4Vcow.

Example 10.3.1

A cue ball weighing 0.17 kg, traveling at 1 m/s, impacts a stationary billiard ball with a mass of 0.15kg as shown below. If the balls collide directly and the collision is elastic, what will the velocities be after the collision (ignore rotational energies)?

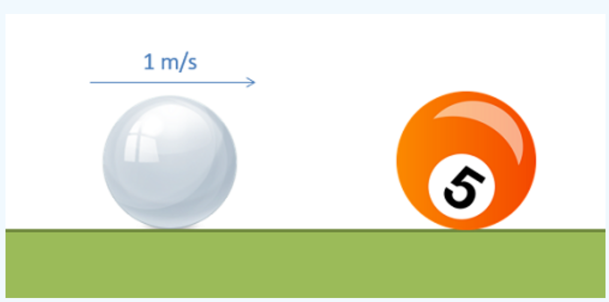

Figure 10.3.2: problem diagram for Example 10.3.1. A cue ball traveling at 1 m/s directly impacts a stationary billiard ball.

Solution

Video 10.3.2: Worked solution to example problem 10.3.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/n1g5RwuFYJI.

Example 10.3.2

A 4.2 g bullet traveling at 965 m/s becomes lodged in a stationary log with a mass of 1.5 kg.

- What is the the velocity of the log and the bullet immediately after the collision?
- What percentage of the kinetic energy was lost in the collision?

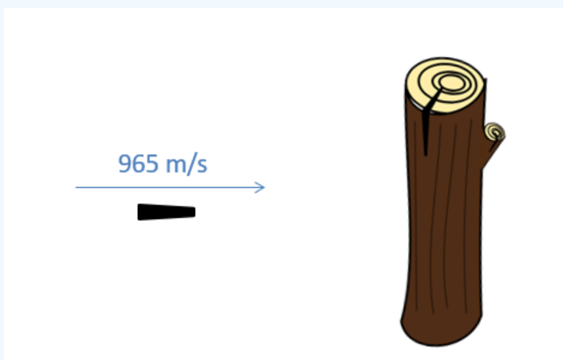

Figure 10.3.3: problem diagram for Example 10.3.2. A bullet traveling at 965 m/s is flying directly towards a large log, where it will become lodged.

Solution

Video 10.3.3: Worked solution to example problem 10.3.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/FyyziesH9LI.

Example 10.3.3

Two masses on a frictionless rod as shown below are set to impact with each other. If the coefficient of restitution between the objects is 0.6, what is the velocity of each body after the collision?

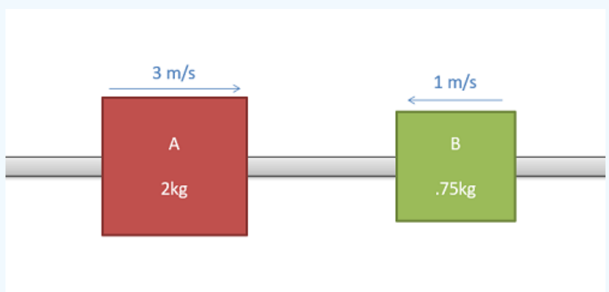

Figure 10.3.4: problem diagram for Example 10.3.3. Two different masses strung on a frictionless rod travel towards each other at different speeds.

Solution

Video 10.3.4: Worked solution to example problem 10.3.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/MOUG_7BqcLY.

10.4: Two-Dimensional Particle Collisions

To analyze collisions in two dimensions, we will need to adapt the methods we used for a single dimension. To start, the conservation of momentum equation will still apply to any type of collision.

$$m_A\,\vec{v}_{A,f}+m_B\,\vec{v}_{B,f}=m_A\,\vec{v}_{A,i}+m_B\,\vec{v}_{B,i} \tag{10.4.1}$$

This is, of course, a vector equation, so we can break all those velocities into components to make our one vector equation into two scalar equations. In the equations below, we break the conservation of momentum equation into x and y components.

$$m_A\,v_{A,f,x}+m_B\,v_{B,f,x}=m_A\,v_{A,i,x}+m_B\,v_{B,i,x} \tag{10.4.2}$$

$$m_A\,v_{A,f,y}+m_B\,v_{B,f,y}=m_A\,v_{A,i,y}+m_B\,v_{B,i,y} \tag{10.4.3}$$

In the subscripts for the velocities, we label the particle (A or B), the pre or post collision state (i or f), and the component (x or y). This triple subscript can make things a bit crowded, but as long as you are methodical about labeling and reading these subscripts it is fairly straightforward. To help ease interpretation, it's recommended that you follow a consistent ordering in the subscripts, labeling body, then pre/post collision, then direction.

To supplement the conservation of momentum equations, we will again need to determine the type of collision, classifying the collision as inelastic (where the two particles stick together after impact) or elastic or semi-elastic (where the particles bounce off of one another).

Inelastic Collisions:

In the case of inelastic collisions, the bodies will have the same final velocity as a consequence of sticking together. Rolling this relationship into the above conservation of momentum equations, we wind up with the following equations. These modified equations are usually enough to solve for the unknowns in the equations.

$$m_{(A+B)}\,v_{f,x}=m_A\,v_{A,i,x}+m_B\,v_{B,i,x} \tag{10.4.4}$$

$$m_{(A+B)}\,v_{f,y}=m_A\,v_{A,i,y}+m_B\,v_{B,i,y} \tag{10.4.5}$$

Elastic and Semi-Elastic Collisions:

Unlike the inelastic collisions, elastic and semi-elastic collisions will have separate velocities for each of the bodies post-collision. With each body having separate x and y components, this represents four unknown variables. Assuming we know all starting conditions, we will need four separate equations to solve for all unknowns.

Unlike the conservation of momentum equation, the conservation of energy equation we would use for elastic collisions is not a vector equation and cannot be broken down into components. Instead we will need to look to the coefficient of restitution, and set it equal to 1 for elastic collisions.

To solve these problems, we will first need to set up a specific set of coordinate axes. These axes will be the tangential direction (along the plane of the collision) and the normal direction (perpendicular to the plane of the collision). An example of these directions is shown in the figure below.

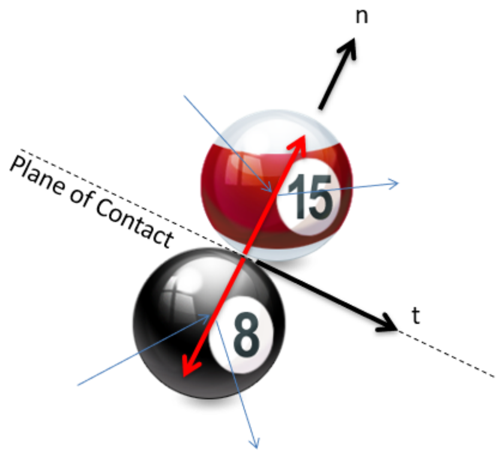

Figure 10.4.1: In a 2-D collision, it is important to identify the normal and tangential directions. The tangential direction will always be along the plane of impact while the normal direction will be perpendicular to the plane of contact.

In examining the figure above, we can see something special about the tangential direction in that there are no forces on either body in this direction. With no forces, there is no impulse, and with no impulse there is no change in momentum for either particle individually in the tangential direction. This means that velocity is conserved for each body in the tangential direction on its own. Adding to that, momentum as a whole is conserved in the normal direction and the coefficient of restitution equation can be applied to the normal direction and we have the four equations we need to solve most problems.

$$v_{A,f,t} = v_{A,i,t} \tag{10.4.6}$$

$$v_{B,f,t} = v_{B,i,t} \tag{10.4.7}$$

$$m_A\, v_{A,f,n} + m_B\, v_{B,f,n} = m_A\, v_{A,i,n} + m_B\, v_{B,i,n} \tag{10.4.8}$$

$$\epsilon = -\frac{v_{A,f,n} - v_{B,f,n}}{v_{A,i,n} + v_{B,i,n}} \tag{10.4.9}$$

To use the above equations, we will need to break all known velocities down into n and t components, then simply plug those values in and solve the above equations. In the end we may also need to convert the found n and t velocity components into x and y components or magnitudes and directions.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/iugmHb5fLMA.

Example 10.4.1

Two cars collide at an intersection as shown below. The cars become entangled with one another, sticking together after the impact. Based on the information given below on the initial velocities and assuming both cars slide away from the crash at the 30-degree angle as shown, what must the initial velocity of car B have been before the impact?

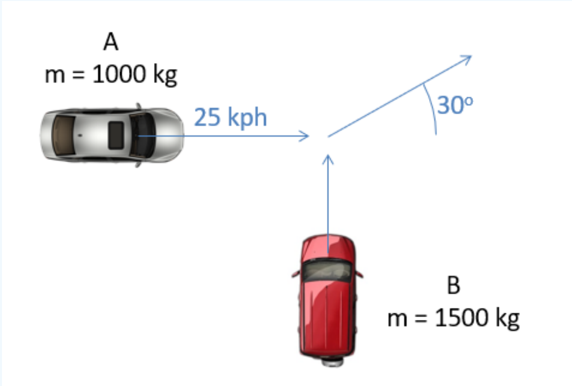

Figure 10.4.2: problem diagram for Example 10.4.1. Two cars of different masses travel towards each other at right angles, colliding and becoming stuck together.

Solution

Video 10.4.2: Worked solution to example problem 10.4.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/RBegCRhcGQc.

Example 10.4.2

Two hockey pucks collide obliquely while sliding on a smooth surface, as shown below. Assume the coefficient of restitution is 0.7 and time of impact is 0.001s.

- What is the final speed of each puck?
- What is the average force exerted on each puck during the impact?

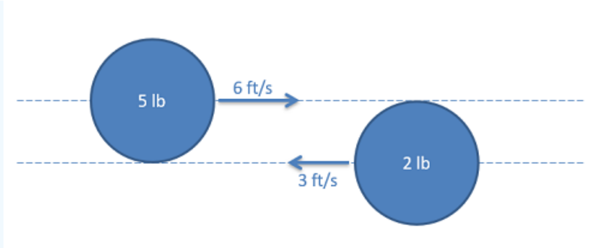

Figure 10.4.3: problem diagram for Example 10.4.2. Two hockey pucks of different masses but the same radius move towards each other, their centers of mass offset by a distance equal to the radius.

Solution

Video 10.4.3: Worked solution to example problem 10.4.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/mRPjLk8RxQw.

10.5: Steady-Flow Devices

A **steady-flow device** is any device that will have a continuous flow of material through it. Some examples of steady-flow devices include pipes, nozzles, diffusers, and pumps. Generally, the material flowing through the device is a gas or liquid, and if the device in any way changes the velocity of the fluid then that fluid will exert a force on the steady flow device in return.

Figure 10.5.1: The nozzle on the fire hose is an example of a steady-flow device. Because the nozzle changes the velocity of the water as it exits the hose, it will take a force to hold the nozzle in place.

In order to determine the forces at play on a steady-flow device, we will start with our impulse-momentum equation.

$$\vec{J} = m\vec{v}_f - m\vec{v}_i \tag{10.5.1}$$

Because this is a continuous process, it doesn't really make sense to have initial and final velocities. Instead, we will have inlet and outlet velocities. Also, the mass will need to be changed to the mass flow rate (the mass entering or leaving the device per unit time) to deal with the continuous-flow nature of the system.

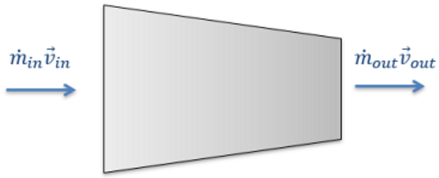

Figure 10.5.2: The mass flow rate is a more appropriate measure for steady-flow devices than standard mass.

Dividing our initial impulse-momentum equation by time on both sides will give us the desired mass flow rate on the right, while the time on the left will cancel out the time component of the impulse.

$$\frac{\vec{F} * t}{t} = \frac{m_{out}}{t}\vec{v}_{out} - \frac{m_{in}}{t}\vec{v}_{in} \tag{10.5.2}$$

Simplifying this equation, we will arrive at our final equation, which relates the force our steady-flow device exerts on the fluid to the mass flow rates and velocities at the inlet and outlet. The force the fluid exerts on the device would simply be equal and opposite to the force below.

$$\vec{F} = \dot{m}_{out}\vec{v}_{out} - \dot{m}_{in}\vec{v}_{in} \tag{10.5.3}$$

One final note is that these equations are vector equations. If the device is changing the direction of the flow of a fluid you will need to break the force and velocities into x and y components and split the above equation into x and y components.

Finding Mass Flow Rate:

If the mass flow rate in or out of your device is not given directly, you may need to find those values. First we can use a simple identity: we know the mass flow rate will be equal to the density of the fluid (ρ) times the volumetric flow rate. Furthermore, the volumetric flow rate can be related to the geometry of the device, in that it will be equal to the average velocity of the fluid at the inlet or outlet times the cross-sectional area at the inlet or outlet. Putting this all together, we arrive at the following formula.

$$\dot{m} = \rho\dot{V} = \rho\vec{v}A \tag{10.5.4}$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/R8A35UeoNyE.

Example 10.5.1

A firefighter supports a hose as shown below. The hose has a volumetric flow rate of 60 gal/min and the nozzle reduces in diameter from 4 cm to 2 cm. What force will the firefighter have to exert, in Newtons, to keep the hose in place?

Figure 10.5.3: A firefighter supporting the nozzle of a hose. Photo by Macomb Paynes, CC BY-NC-SA 2.0.

Solution

Video 10.5.2: Worked solution to example problem 10.5.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/rl4A1N_BNGA.

Example 10.5.2

A 90-degree elbow joint redirects the flow along a pipe of diameter 3 cm. If water (density=1000 kg/m^3) is traveling through the pipe with an average speed of 5 m/s, what is the magnitude and direction of the force the water exerts on the elbow joint?

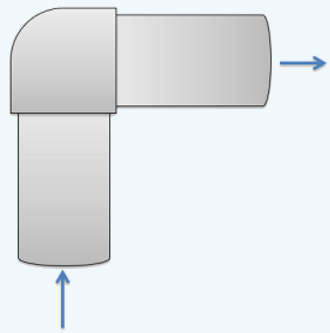

Figure 10.5.4: problem diagram for Example 10.5.2. An inverted-L-shaped assembly of two pipe segments connected by an elbow joint has water entering through the bottom left opening and leaving through the upper right opening.

Solution

Video 10.5.3: Worked solution to example problem 10.5.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/znKq2quYMWk.

10.6: Chapter 10 Homework Problems

Exercise 10.6.1

A jackhammer exerts the impulse shown below on the 1.5-kilogram bit to drive it towards the ground. If the bit starts at rest, what will the expected velocity of the bit be at the end of the impulse?

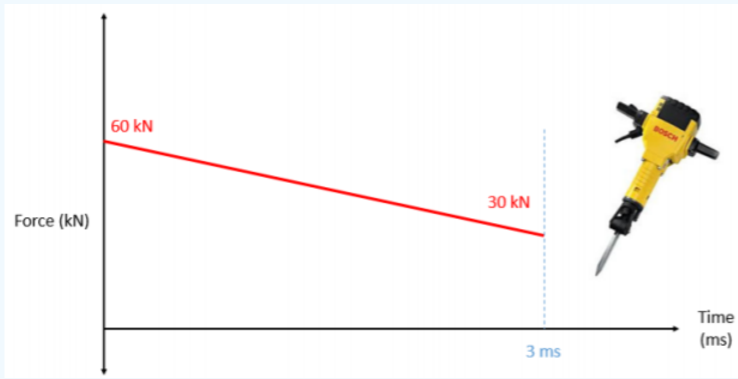

Figure 10.6.1: problem diagram for Exercise 10.6.1. Graph of the force exerted by the jackhammer on its bit as a function of time.

Solution

$v = 90\, m/s$

Exercise 10.6.2

A 0.05-lb arrow traveling at 350 ft/s impacts a 0.4-lb apple on the top of a post that is 3 feet tall. If the arrow becomes lodged in the apple, how far would we expect the apple to travel (d) before hitting the ground?

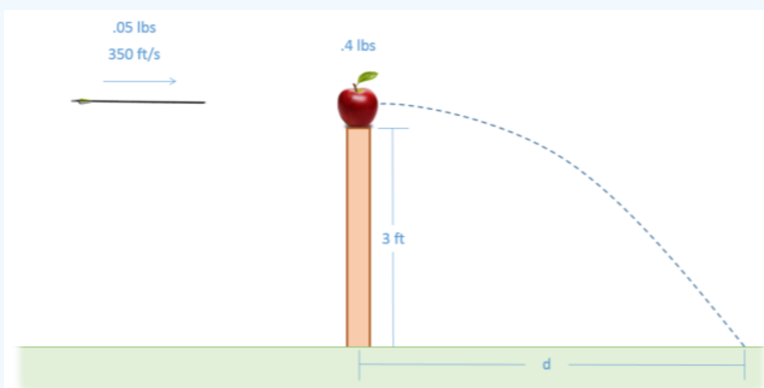

Figure 10.6.2: problem diagram for Exercise 10.6.2. An arrow is about to hit an apple; the two will fall together off the pole where the apple is balanced.

Solution

$d = 16.8\, ft$

Exercise 10.6.3

A basketball impacts a metal surface as shown below. If the initial velocity of the basketball was 3 ft/s straight down and the coefficient of restitution is 0.85, what is the expected speed and direction (θ) of the ball after the impact?

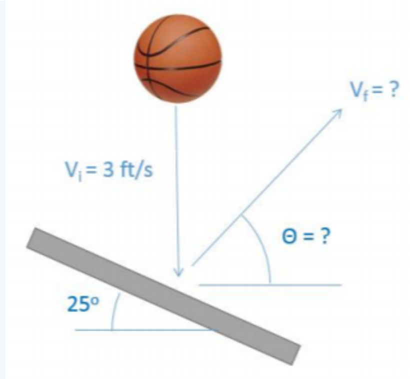

Figure 10.6.3: problem diagram for Exercise 10.6.3. A basketball falls onto a metal surface tilted at 25° from the horizontal, bouncing off at some speed at an angle of θ from the horizontal.

Solution

$v = 2.64\,ft$, $\theta = 36.25^\circ$

Exercise 10.6.4

Puck A, traveling with an initial velocity of 5 m/s, strikes the stationary Puck B. Assuming the collision is elastic, what will the velocity of each puck be immediately after the collision?

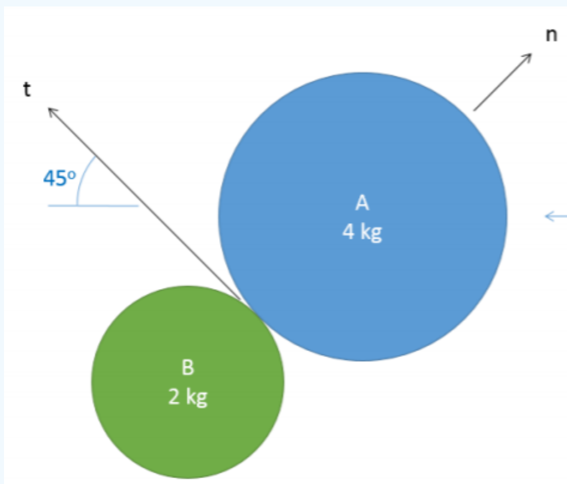

Figure 10.6.4: problem diagram for Exercise 10.6.4. Puck A moves towards the left, striking the upper right corner of Puck B and creating a plane of contact that translates to a tangential axis 45° from the horizontal.

Solution

$\vec{v}_{A,f} = [-3.34, 1.67]\,m/s$

$\vec{v}_{B,f} = [-3.34, -3.34]\,m/s$

Exercise 10.6.5

A jet engine with a mass of 700 kg and an air mass flow rate of 50 kg/s is mounted to a stand as shown below (a set of legs on each side, only one half shown). Based on the input and output velocities shown below, determine the thrust force of the engine and the forces in stand members AB, AD, and CD. Be sure to indicate if each member is in tension or compression.

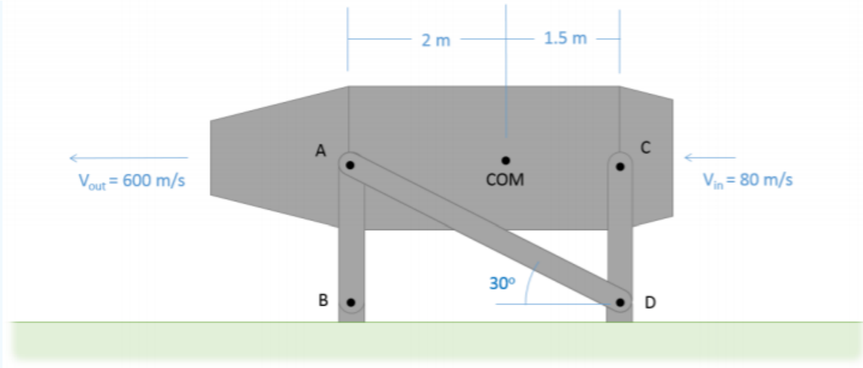

Figure 10.6.5: problem diagram for Exercise 10.6.5. A jet engine facing towards the left is mounted on a stand consisting of 4 legs, two on the side of the engine facing the viewer and two on the opposite side, with a single diagonal member connecting the legs within each set.

Solution

$F_{thrust} = 26\,kN$

$F_{AB} = 6.04\,kN\,T$

$F_{AD} = 15.01\,kN\,C$

$F_{CD} = 1.96\,kN\,C$

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CHAPTER OVERVIEW

11: Rigid Body Kinematics

11.1: Fixed-Axis Rotation in Rigid Bodies

When moving from particle kinematics to **rigid body** kinematics, we add the **rotation** of a body into the motion analysis process. Some bodies will translate and rotate at the same time, but many engineered systems have components that simply rotate about some fixed axis. We will start our examination of rigid body kinematics by examining these fixed-axis rotation problems, where rotation is the only motion we need to worry about.

Figure 11.1.1: The flywheel on this antique motor is a good example of fixed axis rotation.

Figure 11.1.2: The rotating x-ray tube within the gantry of this CT machine is another example of fixed axis rotation. Image by Thirteen of Clubs, CC-BY-SA 2.0.

Angular Position, Velocity, and Acceleration:

Just as with translational motion, we will have angular positions which we can take the derivative of to find angular velocities, which we can again take the derivative of to find angular accelerations. Since we can only have a single axis of rotation in two-dimensional problems (rotating about the z-axis, with counterclockwise rotations being positive, and clockwise rotations being negative) the equations will mirror the one-dimensional equations used in particle kinematics.

$$\text{Angular Position:} \quad \theta(t) \tag{11.1.1}$$

$$\text{Angular Velocity:} \quad \omega(t)=\frac{d\theta}{dt}=\dot{\theta} \tag{11.1.2}$$

$$\text{Angular Acceleration:} \quad \alpha(t)=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}=\ddot{\theta} \tag{11.1.3}$$

Also as with one-dimensional translational motion, we can use integration to move in the opposite direction (just remember your constants of integration).

$$\text{Acceleration:} \quad \alpha(t) \tag{11.1.4}$$

$$\text{Velocity:} \quad \omega(t)=\int \alpha(t)\,dt \tag{11.1.5}$$

$$\text{Position:} \quad \theta(t)=\int \omega(t)\,dt=\int\int \alpha(t)\,dt\,dt \tag{11.1.6}$$

If we have constant angular accelerations, we can also use the following formulas adapted from one-dimensional motion.

$$\text{Acceleration:} \quad \alpha(t)=\alpha \tag{11.1.7}$$

$$\text{Velocity:} \quad \omega(t)=\alpha t+\omega_0 \tag{11.1.8}$$

$$\text{Position:} \quad \theta(t)=\frac{1}{2}\alpha t^2+\omega_0 t+\theta_0 \tag{11.1.9}$$

$$\text{Without time:} \quad \omega^2-\omega_0^2=2\alpha(\theta-\theta_0) \tag{11.1.10}$$

Velocity and Acceleration of a Point on a Rotating Body:

With fixed-axis rotation, there is a single point on a body that does not move; however, all other points on this body will have some velocity and some acceleration due to the rotation of the body itself.

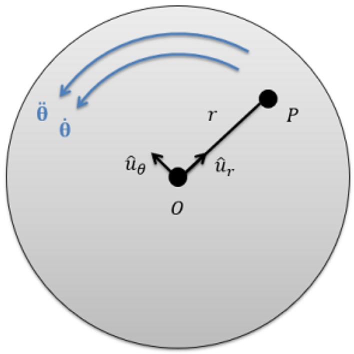

Figure 11.1.3: The diagram above shows a point P on a body rotating about fixed axis O. Point P is a constant distance r away from the fixed point O. We will adapt the polar kinematics equations to find the velocity and acceleration of point P at any given instant.

To determine the velocities and accelerations of these points, we will adapt the equations we used for polar coordinates. As a reminder, these equations were as follows:

$$\text{Velocity:} \quad v=\dot{r}\hat{u}_r+r\dot{\theta}\hat{u}_\theta \tag{11.1.11}$$

$$\text{Acceleration:} \quad a=(\ddot{r}-r\dot{\theta}^2)\hat{u}_r+(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{u}_\theta \tag{11.1.12}$$

To simplify the above equations, we can note that for a rigid body, the point P never gets any closer or further away from the fixed center point O. This means that the distance r never changes, and the $\dot{r}$ and $\ddot{r}$ terms in the above equations are zero. Putting this to work, we can simplify the above equations into the equations below.

$$\text{Velocity:} \quad v=r\dot{\theta}\hat{u}_\theta \tag{11.1.13}$$

$$\text{Acceleration:} \quad a=(-r\dot{\theta}^2)\hat{u}_r+(r\ddot{\theta})\hat{u}_\theta \tag{11.1.14}$$

These equations allow us to find the velocity and acceleration of any point on a body rotating about a fixed axis, given that we know the angular velocity of the body \((\dot{\theta})\), the angular acceleration of the body $(\ddot{\theta})$, and the distance from the point to the axis of rotation (r).

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/X02-jN5cwF4.

Example 11.1.1

A flywheel rotates on a fixed axle in a steam engine. The flywheel is rotating at a rate of 600 rpm before a brake begins decelerating the flywheel at a constant rate of 30 rad/s^2. What is the time required to bring the flywheel to a complete stop? How many rotations does the flywheel go through while decelerating?

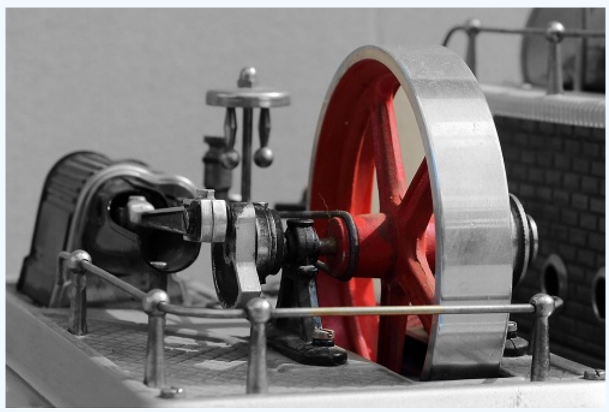

Figure 11.1.4: A steam engine's flywheel rotates on a fixed axle.

Solution

Video 11.1.2: Worked solution to example problem 11.1.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/z1S9O0XFkDE.

Example 11.1.2

A hard drive platter 8 cm in diameter is rotating at a constant rate of 3600 rpm. What is the velocity of a point on the outer edge of the platter? What is the acceleration experienced by a point on the edge of the platter?

Figure 11.1.5: A hard drive platter rotates with its center fixed in place.

Solution

Video 11.1.3: Worked solution to example problem 11.1.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/n4OixdyqqoU.

11.2: Belt- and Gear-Driven Systems

Belt-and-pulley systems, along with gear-driven systems, represent the common ways that engineers transfer rotational motion and torque from one shaft to another shaft. Belts offer flexibility in that the shafts do not need to be right next to one another, and gears are more commonly used in high-load applications.

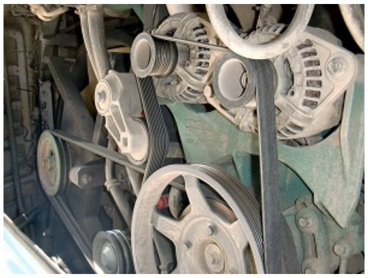

Figure 11.2.1: Belts and pulleys are often used to transmit motion and torque from one shaft to another.

Figure 11.2.2: Gears are another common way of transmitting motion and torque from one shaft to another shaft.

Position, Velocity, and Acceleration in Belt-Driven Systems

The diagram below shows a simple belt-driven system. Pulley A and Pulley B each have their own radius, and are connected via a belt that we will assume is not slipping relative to the pulleys. Each pulley is undergoing fixed axis rotation and will therefore follow those kinematic rules separately; however, the motion of the belt can be used to relate the motion of the two pulleys.

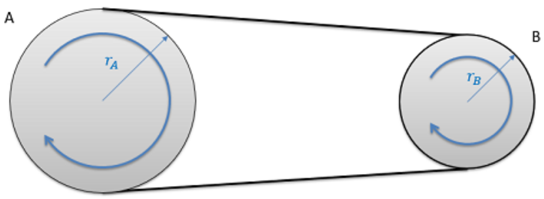

Figure 11.2.3: The diagram above shows a simple belt driven system connecting pulley A and pulley B.

As a constraint, we can assume the speed of the pulley will be uniform throughout the whole loop at any one time. If this was not true, the belt would be bunching up in some locations and stretching out in other areas. If the belt isn't slipping, the speed of the belt will be the same as the speed of the edge of each of the two pulleys. Setting these two speeds equal to one another and working backwards to relate them to angular velocities, we wind up with the middle equation below. Taking the integral or derivative allows us to also relate angular displacements or angular accelerations with similar equations.

$$\text{Angular Displacements:} \quad r_A(\Delta\theta_A) = r_B(\Delta\theta_B) \tag{11.2.1}$$

$$\text{Angular Velocities:} \quad r_A\omega_A = r_B\omega_B \tag{11.2.2}$$

$$\text{Angular Accelerations:} \quad r_A\alpha_A = r_B\alpha_B \tag{11.2.3}$$

If we have a more complex series of belts and pulleys, we will analyze the system one step at at time. This will include pulleys connected via belts as we had above, as well as pulleys connected via a shaft as shown with pulleys B and C in the diagram below.

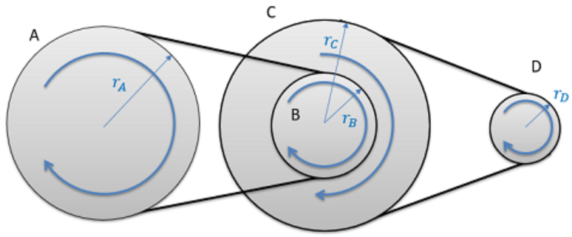

Figure 11.2.4: The diagram above shows a multi-stage belt driven system connecting pulley A and pulley D. Pulleys A and B are connected via a belt, then B and C are on the same shaft, then C and D are connected via a belt.

With pulleys on the same shaft, the angular displacements, the angular velocities, and the angular accelerations will all be the same.

$$\Delta\theta_B = \Delta\theta_C \ ; \quad \omega_B = \omega_B \ ; \quad \alpha_B = \alpha_C \tag{11.2.4}$$

If we know the angular displacement, angular velocity, or angular acceleration of pulley A, we could find the angular displacement, angular velocity, or angular acceleration of pulley D by moving one interaction at a time (finding the motion of pulley B, then C, then D).

Position, Velocity, and Acceleration in Gear Systems:

The diagram below shows a simple gear system. Gear A and Gear B each have their own radius, and are interacting at their point of contact. Each gear is undergoing fixed-axis rotation and will therefore follow those kinematic rules separately; however, the motion of the teeth at the point of contact can be used to relate the motion of one gear to the next.

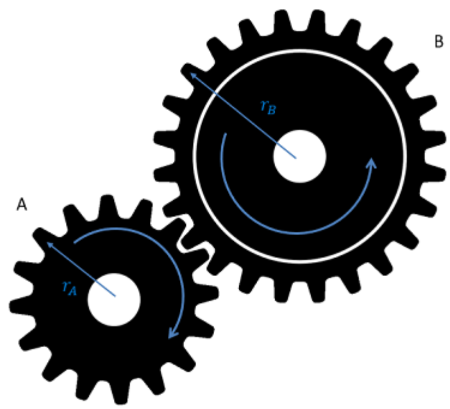

Figure 11.2.5: The diagram shows a simple gear system with gears A and B interacting.

As a constraint, we can assume that the speed of the teeth at the point of contact will be the same. If this were not true, the teeth of one gear would be passing through the teeth of the other gear. Setting these these two speeds equal to one another and working backwards to relate the angular velocities, we find the second equation below. Taking the integral or derivative allows us to also relate the angular displacements or angular accelerations with similar equations.

$$\text{Angular Displacements:} \quad r_A(\Delta\theta_A) = -r_B(\Delta\theta_B) \tag{11.2.5}$$

$$\text{Angular Velocities:} \quad r_A\omega_A = -r_B\omega_B \tag{11.2.6}$$

$$\text{Angular Accelerations:} \quad r_A\alpha_A = -r_B\alpha_B \tag{11.2.7}$$

You will notice that the equations above match the equations we had for belt-driven systems, except for the minus sign on the right side of each equation. This is because meshed gears rotate in opposite directions (if one gear rotates clockwise, the other will rotate counterclockwise), while pulleys in belt-driven systems always rotate in the same direction.

Also similar to belt driven systems, we can have compound gear trains with three or more gears similar to the figure below. In these scenarios, we will also likely have gears that are connected via a shaft like the blue and yellow gears shown below. In such situations, the gears on the same shaft will have matching angular displacements, angular velocities, and angular accelerations. As with belt driven systems, you simply need to take the gear train one step at a time, applying the right set of equations to match each step in the interaction.

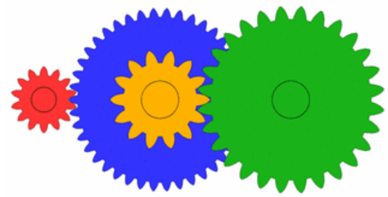

Figure 11.2.6: The animated diagram above shows a compound gear train. The red and blue gears interact via meshing teeth, then the blue and yellow gears are on the same shaft, then finally the yellow and green gears interact via meshing teeth.

A concept that is commonly used in gear trains that is not commonly used in belt driven systems is the concept of the **gear ratio**. For any gear train, the gear ratio is defined as the angular speed of the input divided by the angular speed of the output. Based on the equations above, we can also prove that the ratio of angular displacements or angular accelerations will similarly be equal to the gear ratio. However, the gear ratio is always defined as a positive number, so you will still need to use intuition to determine the direction of output.

$$\text{Gear Ratio} = \frac{\omega_{input}}{\omega_{output}} = \frac{\Delta\theta_{input}}{\Delta\theta_{output}} = \frac{\alpha_{input}}{\alpha_{output}} \tag{11.2.8}$$

In a simple two-gear system, the gear ratio will be equal to the radius of the output gear divided by the radius of the input gear, or the number of teeth on the output gear divided by the number of teeth on the input gear (since the number of teeth will be directly proportional to the radius). In compound gear trains this simple calculation given below will not work, but if you are given the gear ratio for a compound gear train, you can still apply the equations above.

$$\text{Gear Ratio} = \frac{\omega_{input}}{\omega_{output}} = \frac{r_{output}}{r_{input}} = \frac{N_{output}}{N_{input}} \tag{11.2.9}$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/6AoVrO5s7ME.

Example 11.2.1

If the input pulley A as shown below is rotating at a rate of 10 rad/s, what is the speed of the output pulley at D? How many rotations does D go through in the time it takes for A to make one full rotation?

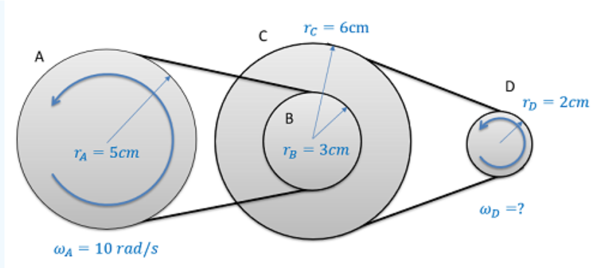

Figure 11.2.7: problem diagram for Example 11.2.1. Pulleys A and B are connected by one belt; pulley C, which is on the same shaft as B, is connected to pulley D by a second belt.

Solution

Video 11.2.2: Worked solution to example problem 11.2.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/IxhXp7eBx3k.

Example 11.2.2

A car is moving at 40 ft/s on 18-inch-diameter wheels. What is the angular velocity of the wheels on the car? If the car is in third gear with a gear ratio of 4.89:1, what is the angular velocity of the engine in rotations per minute? (Hint: the engine is the input to the gear train and the wheels are the output of the gear train.)

Figure 11.2.8: A police car driving on a highway.

Solution

Video 11.2.3: Worked solution to example problem 11.2.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/-2Gta-grqoI.

11.3: Absolute Motion Analysis

Absolute motion analysis is one method used to analyze bodies undergoing **general planar motion**. General planar motion is motion where bodies can both translate and rotate at the same time. Besides absolute motion analysis, the alternative is relative motion analysis. Either method can be used for any general planar motion problem, but one method may be significantly easier to apply for a given situation.

Absolute motion analysis will require calculus, and is generally faster for simple problems and problems where only the velocities (and not accelerations) are required. Relative motion analysis will not require calculus, but does necessitate using multiple coordinate systems; it is generally easier to use for more complex problems and problems where velocities and accelerations are being analyzed.

Utilizing Absolute Motion Analysis:

To start our discussion on absolute motion analysis, we are going to imagine a simple robotic arm such as the one below. In this arm, we have two arm sections of fixed length with motors causing rotations at joint A and joint B.

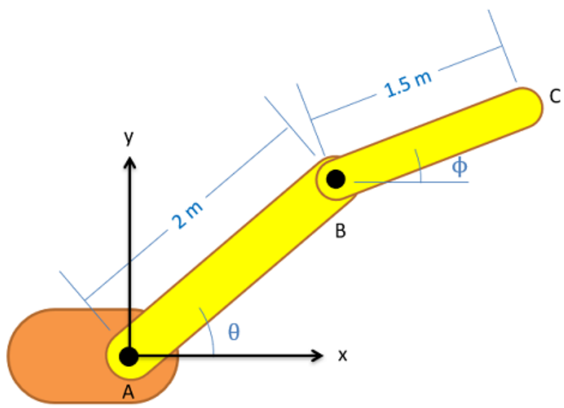

Figure 11.3.1: This robotic arm has a fixed base at A, and two fixed length arm sections (AB and BC) that are controlled via motors at joints A and B. The end effector of the robotic arm is at C.

The first step in absolute motion analysis is come up with a set of equations describing the position of some point of interest. In this case we will be looking at the position of the end effector of the arm at point C, and we will write an equation for the x position and the y position of this point with respect to the fixed origin point at A. In these equations, anything that is a constant (such as the length of the arm pieces) can be put in as a number, but anything that will change, such as angles θ and ϕ, will need to remain as variables in these equations even if they are known at the moment. Using the values in the diagram, we would wind up with the following two position equations.

$$x\text{-position:} \quad x_C = 2\cos(\theta) + 1.5\cos(\phi) \tag{11.3.1}$$

$$y\text{-position:} \quad y_C = 2\sin(\theta) + 1.5\sin(\phi) \tag{11.3.2}$$

To find the velocity of point C in the x and y directions, we simply need to take the derivatives of the position equations. The velocity equations for our robotic arm are below.

$$x\text{-velocity:} \quad v_{x\,C} = -2\sin(\theta)\,\dot{\theta} - 1.5\sin(\phi)\,\dot{\phi} \tag{11.3.3}$$

$$y\text{-velocity:} \quad v_{y\,C} = 2\cos(\theta)\,\dot{\theta} + 1.5\cos(\phi)\,\dot{\phi} \tag{11.3.4}$$

To find the acceleration of point C in the x and y directions, we simply need to take the derivatives of the velocity equations. The acceleration equations for our robotic arm in the x and y directions are shown below.

$$x \text{ acceleration:} \quad a_{x\,C} = -2\cos(\theta)\,\dot{\theta}^2 - 2\sin(\theta)\,\ddot{\theta} - 1.5\cos(\phi)\,\dot{\phi}^2 - 1.5\sin(\phi)\,\ddot{\phi} \tag{11.3.5}$$

$$y \text{ acceleration:} \quad a_{y\,C} = -2\sin(\theta)\,\dot{\theta}^2 + 2\cos(\theta)\,\ddot{\theta} - 1.5\sin(\phi)\,\dot{\phi}^2 + 1.5\cos(\phi)\,\ddot{\phi} \tag{11.3.6}$$

Once we have the velocity and acceleration equations, we can start solving for any unknowns. If we have known angular velocities and accelerations ($\dot{\theta}$, $\ddot{\theta}$, $\dot{\phi}$, and $\ddot{\phi}$) we can plug those in to find the velocity and acceleration vectors for the end effector. In other instances, we may known the desired motion of the end effector (v_{xC}, v_{yC}, a_{xC}, and a_{yC}) and will have to plug those values into the equations to solve for unknowns such as $\dot{\theta}$ and so on.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/y1Pq2VP3qas.

Example 11.3.1

The robotic arm shown below has a fixed orange base at A and fixed-length members AB and BC. Motors at A and B allow for rotational motion at the joints. Based on the angular velocities and accelerations shown at each joint, determine the velocity and the acceleration of the end effector at C.

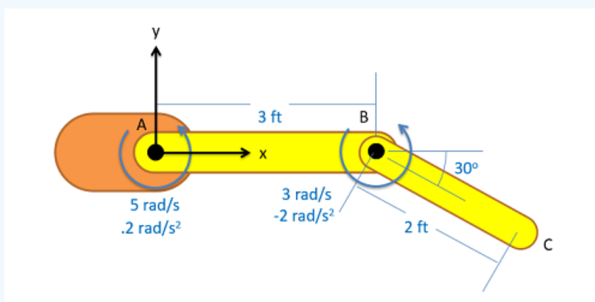

Figure 11.3.2: problem diagram for Example 11.3.1. A robotic arm with two segments has motors at its joints, rotating at known velocities and accelerations.

Solution

Video 11.3.2: Worked solution to example problem 11.3.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/ZBj17t8mhZc.

Example 11.3.2

The robotic arm from the previous problem is in the configuration shown below. Assume that θ is currently 30 degrees and that point C currently lies along the x axis. If we want the end effector at C to travel 1 ft/s in the negative x-direction, what should the angular velocities be at joints A and B?

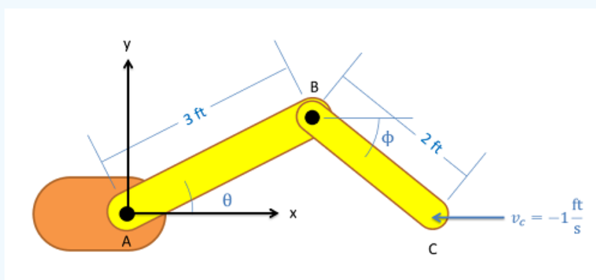

Figure 11.3.3: problem diagram for Example 11.3.2. A two-segmented robotic arm has an effector on its free end moving at a known speed, as a result of the rotation of the motors at the two joints.

Solution

Video 11.3.3: Worked solution to example problem 11.3.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/O5mtTTpK1RQ.

Example 11.3.3

A ladder is propped up against a wall as shown below. If the base of the ladder is sliding out at a speed of 2 m/s, what is the speed of the top of the ladder?

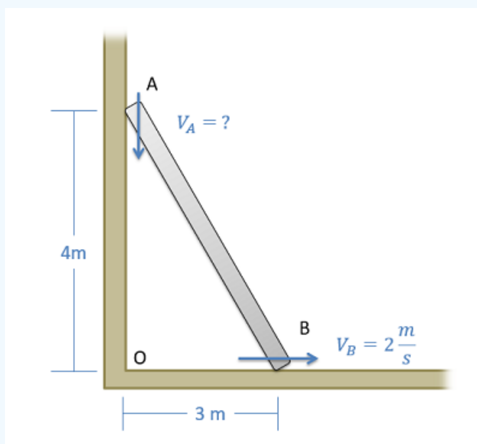

Figure 11.3.4: problem diagram for Example 11.3.3. A ladder propped against a wall is falling, as its foot slides away from the wall.

Solution

Video 11.3.4: Worked solution to example problem 11.3.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/fOfEaFUiDZI.

Example 11.3.4

The crank-rocker mechanism as shown below consists of a crank with a radius of 0.5 meters rotating about its fixed center at C, at a constant rate of 2 rad/s clockwise. Rocker AB is fixed at its base at A and connects to point B along the edge of the crank. The pin at point B can slide along a frictionless slot in AB. In the current state, what is the angular velocity of rocker AB?

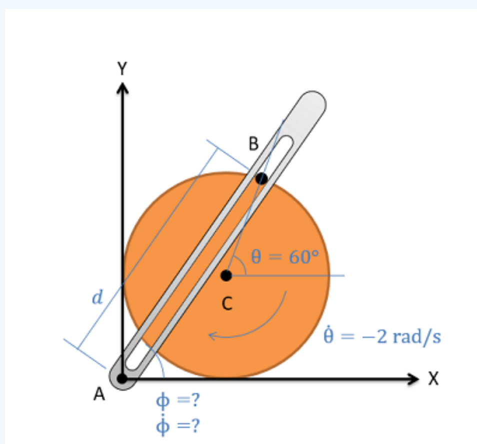

Figure 11.3.5: problem diagram for Example 11.3.4. A circular side view of a rotating crank with a pin in one edge. The pin moves through a slot at one end of a rocker bar, whose other end is fixed in place.

Solution

Video 11.3.5: Worked solution to example problem 11.3.4, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/DydbOzigdpU.

11.4: Relative Motion Analysis

Relative motion analysis is one method used to analyze bodies undergoing **general planar motion**. General planar motion is motion where bodies can both translate and rotate at the same time. Besides relative motion analysis, the alternative is absolute motion analysis. Either method can be used for any general planar motion problem, however one method may be significantly easier to apply in certain situations.

Absolute motion analysis will require calculus and is generally faster for simple problems and problems where only the velocities (and not accelerations) are required. Relative motion analysis will not require calculus, but does necessitate using multiple coordinate systems and is generally easier to use for more complex problems and problems where velocities and accelerations are being analyzed.

Utilizing Relative Motion Analysis

To start our discussion on relative motion analysis, we are going to imagine a simple robotic arm such as the one below. In this arm, we have two arm sections of fixed length, with motors causing rotations at joint A and joint B.

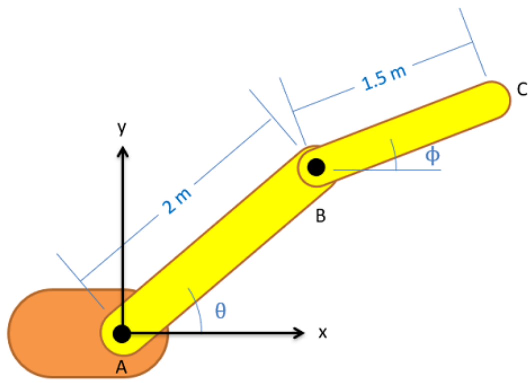

Figure 11.4.1: This robotic arm has a fixed base at A, and two fixed length arm sections (AB and BC) that are controlled via motors at joints A and B. The end effector of the robotic arm is at C.

The first step in relative motion analysis is to break the motion down into simple steps and assign a coordinate system (with r and θ directions) to each step in the chain of motion. We will always start at a fixed point and move step by step from there. In the case of our robotic arm, Joint A is the only point that will not be moving so we start there; then we have two cases of rotation without extension as we move from A to B, then B to C.

Since there are two steps to the motion, there will be two coordinate systems. The first coordinate system will be attached to member AB, with the r direction going from point A to point B. The θ direction will then be ninety degrees counterclockwise from the r direction. The second coordinate system will be attached to member BC, with the r direction going from point B to point C. Again, the θ direction is ninety degrees counterclockwise from the r direction. At this point it is usually good to identify the angles of each of the r and θ directions with respect to ground. Below is a picture of the robotic arm with both coordinate systems drawn in.

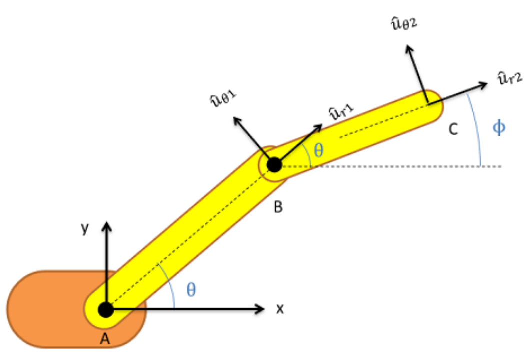

Figure 11.4.2: Two coordinate systems are used in the relative motion analysis of this robotic arm. The first coordinate system is attached to member AB while the second is attached to member BC. The r direction will always line up with the two endpoints of the member, while the θ direction will always be 90° counterclockwise from the corresponding r direction.

For relative motion analysis, we can identify the velocity or acceleration of the end point of the arm (C) with respect to the ground (A) by finding the velocity acceleration of B with respect to A and adding the velocity or acceleration of C with respect to B, just as we did with particles.

$$\text{Velocity:} \quad \vec{v}_{C/A} = \vec{v}_{B/A} + \vec{v}_{C/B} \tag{11.4.1}$$

$$\text{Acceleration:} \quad \vec{a}_{C/A} = \vec{a}_{B/A} + \vec{a}_{B/C} \tag{11.4.2}$$

The relative motion analysis equations above are for a two-part motion (as there are two sections to the arm in our example), but we can easily expand the above equation into three, four, or even more pieces for more complex mechanisms by adding more steps to the left side of our equation.

To use the above equations, we will need to plug in the information we know. Plugging in velocities or accelerations that are given as part of the problem for any particular points is a good place to start. If the velocity of points B or C were given, for example, we would plug that in for $\vec{v}_{B/A}$ or $\vec{v}_{C/A}$ respectively. Do remember that point A is our fixed ground point, so $\vec{v}_{C/A}$ is the velocity of point C relative to the ground while $\vec{v}_{C/B}$ is the velocity of point C with respect to point B. If any point is fixed (other than our original ground point, which will always be fixed), we can also plug in zeros for both velocity and acceleration of that point. Remember that this equation is a vector equation, so these velocities have both a magnitude and a direction.

To take into account rotation or extension of any individual pieces, we will need to look back to our kinematics equations for polar coordinator systems. Below are the equations we had for velocity and acceleration.

$$\text{Velocity:} \quad v = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta \tag{11.4.3}$$

$$\text{Acceleration:} \quad a = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta \tag{11.4.4}$$

In the above equations, r represents the length of the respective arm piece (the length of member AB or BC), $\dot{r}$ represents the rate at which that length is increasing, and $\ddot{r}$ represents the rate at which that rate is increasing. The θ term represents current angle of the arm piece, $\dot{\theta}$ is the angular velocity of that arm piece, and $\ddot{\theta}$ is the angular acceleration of that arm piece. The u_r and u_θ directions are the r and θ directions that are attached to that particular section of the arm (in our earlier drawing, $\hat{u}_{r_1}$ and $\hat{u}_{\theta_1}$ were attached to member AB and $\hat{u}_{r_2}$ and $\hat{u}_{\theta_2}$ were attached to member BC).

Though it's certainly possible to have a mechanism that is rotating and extending at the same time, we will often have either just simple rotation, or just simple extension along a fixed direction. With **simple rotation**, the $\dot{r}$ and $\ddot{r}$ terms are zero, and our equations reduce down to the following.

$$\text{Velocity:} \quad v = r\dot{\theta}\hat{u}_\theta \tag{11.4.5}$$

$$\text{Acceleration:} \quad a = -r\dot{\theta}^2\hat{u}_r + r\ddot{\theta}\hat{u}_\theta \tag{11.4.6}$$

If we have **simple extension**, where the length of the piece is changing but there is no rotation, the $\dot{\theta}$ and $\ddot{\theta}$ terms would be zero and our original equations would reduce down to the following.

$$\text{Velocity:} \quad v = \dot{r}\hat{u}_r \tag{11.4.7}$$

$$\text{Acceleration:} \quad a = \ddot{r}\hat{u}_r \tag{11.4.8}$$

If we use the appropriate set of equations for the type of motion in each step and plug in known quantities for angular velocities, angular accelerations, and rates or extension for each piece, we can add these pieces to our relative motion analysis equation from earlier. Again, these are vectors so be sure to indicate both the magnitudes and directions when we put them into the equation.

Once we have everything in our vector equation, we will break the vector equation into x and y components in order to solve for any unknowns. Simply find the angles or each of the r and θ directions using your diagram and then use sines and cosines to break the individual vectors down into x and y components. Once you have everything in component form, you should be able to solve for any unknowns in your equations. As a note, it is often necessary to start with the velocity equations and solve for some unknowns there before moving on to the acceleration equations.

Alternate Notation for Rigid Body Relative Motion

In some cases, we want to analyze multiple points on the same or several rigid bodies. Another notation used for this is:

$$\text{Position:} \quad \vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \tag{11.4.9}$$

$$\text{Velocity:} \quad \vec{v}_B = \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \tag{11.4.10}$$

$$\text{Acceleration:} \quad \vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \tag{11.4.11}$$

In this notation, A and B are two points on the same rigid body. When we use only one subscript, that indicates the position/velocity/acceleration of that point with respect to the fixed coordinate system. Because the body is rigid, the two points will not change the distance between them, but the position vector between them can change orientation (which is where the relative velocity between the two points comes from). $\vec{\omega}$ is the angular velocity of that rigid body (analogous to $\dot{\theta}$), and $\vec{\alpha}$ is the angular acceleration of that rigid body (analogous to $\ddot{\theta}$).

For planar motion, where the angular velocity vector (out of plane) is always perpendicular to the position vector (in the plane), the acceleration can be simplified to:

$$\text{Acceleration:} \quad \vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \tag{11.4.12}$$

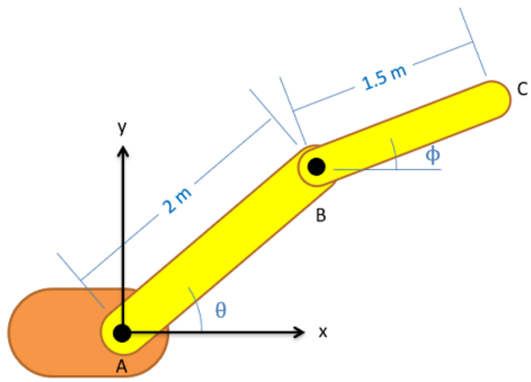

Figure 11.4.1, this robotic arm has a fixed base at A, and two fixed length arm sections (AB and BC) that are controlled via motors at joints A and B. The end effector of the robotic arm is at C.

If we consider the same robotic arm, we can translate our previous notation to the new rigid body notation:

$$\dot{\theta}\hat{k} = \vec{\omega}_{AB} \tag{11.4.13}$$

$$\dot{\phi}\hat{k} = \vec{\omega}_{BC} \tag{11.4.14}$$

$$\ddot{\theta}\hat{k} = \vec{\alpha}_{AB} \tag{11.4.15}$$

$$\ddot{\phi}\hat{k} = \vec{\alpha}_{BC} \tag{11.4.16}$$

All vectors will be defined with respect to the x and y coordinates in the coordinate system shown, rather than the radial and theta directions as above.

If we want to find the velocity and acceleration of the end point, C, knowing the angular velocities and accelerations, we will find the following relationships:

$$\text{Velocity at point B:} \quad \vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \tag{11.4.17}$$

$$\text{Velocity at point C:} \quad \vec{v}_C = \vec{v}_B + \vec{\omega} \times \vec{r}_{C/B} = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} + \vec{\omega} \times \vec{r}_{C/B} \tag{11.4.18}$$

$$\text{Acceleration at point B:} \quad \vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \tag{11.4.19}$$

$$\text{Acceleration at point C:} \quad \vec{a}_C = \vec{a}_B + \vec{\alpha} \times \vec{r}_{C/B} - \omega^2 \vec{r}_{C/B} = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} + \vec{\alpha} \times \vec{r}_{C/B} - \omega^2 \vec{r}_{C/B} \tag{11.4.20}$$

Video lecture covering this chapter, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/6qAY9iB9xi0.

Example 11.4.1

The robotic arm shown below has a fixed orange base at A and fixed length members AB and BC. Motors at A and B allow for rotational motion at the joints. Based on the angular velocities and accelerations shown at each joint, determine the velocity and the acceleration of the end effector at C.

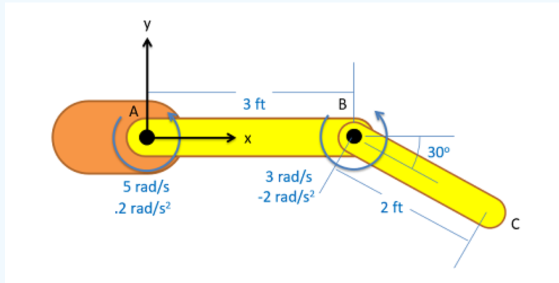

Figure 11.4.4: problem diagram for Example 11.4.1. A robotic arm with two segments has motors at its joints, rotating at known velocities and accelerations.

Solution

Video 11.4.2: Worked solution to example problem 11.4.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/fOjS1O2-3LQ.

Example 11.4.2

The robotic arm from the previous problem is in the configuration shown below. Assume that θ is currently 30 degrees and that point C currently lies along the x axis. If we want the end effector at C to travel 1 ft/s in the negative x-direction, what should the angular velocities be at joints A and B?

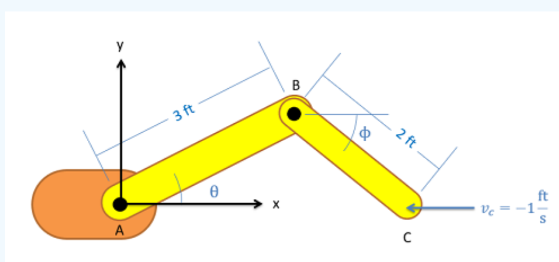

Figure 11.4.5: problem diagram for Example 11.4.2. A two-segmented robotic arm has an effector on its free end moving at a known speed, as a result of the rotation of the motors at the two joints.

Solution

Video 11.4.3: Worked solution to example problem 11.4.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/esp_gz8VKRs.

Example 11.4.3

A ladder is propped up against a wall as shown below. If the base of the ladder is sliding out at a speed of 2 m/s, what is the speed of the top of the ladder?

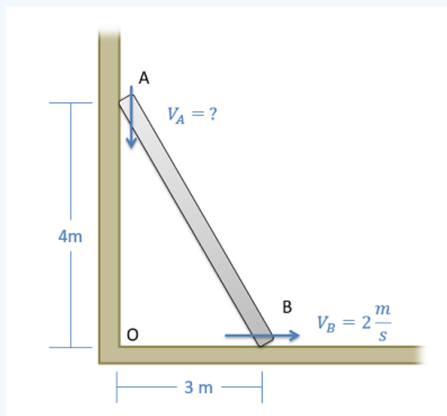

Figure 11.4.6: problem diagram for Example 11.4.3. A ladder propped against a wall is falling, as its foot slides away from the wall.

Solution

Video 11.4.4: Worked solution to example problem 11.4.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/SzAq3HD7qJY.

Example 11.4.4

The crank-rocker mechanism as shown below consists of a crank with a radius of 0.5 meters rotating about its fixed center at C, at a constant rate of 2 rad/s clockwise. Rocker AB is fixed at its base at A and connects to point B along the edge of the crank. The pin at point B can slide along a frictionless slot in AB. In its current state, what is the angular velocity of rocker AB?

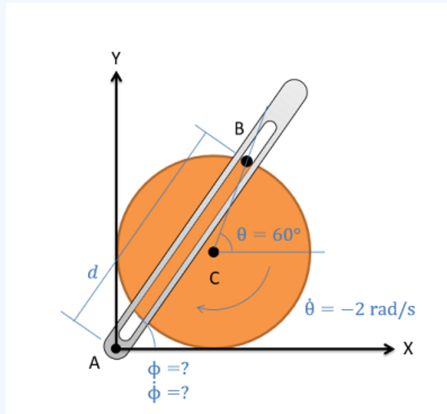

Figure 11.4.7: problem diagram for Example 11.4.4. A circular side view of a rotating crank with a pin in one edge. The pin moves through a slot at one end of a rocker bar, whose other end is fixed in place.

Solution

Video 11.4.5: Worked solution to example problem 11.4.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Exys74nj02I.

11.5: Rotating Frame Analysis

Rotating frame analysis is a specialized part of relative motion analysis. It is typically performed in Cartesian (x-y) coordinates for rigid bodies. In the previous section, rotating frames in polar coordinates were used to solve problems with formulae similar to those in particle kinematics. We will adopt a slightly different notation that is specialized to rigid bodies. These formulae are the most general planar kinematics formulae - that is, they can always be used, and will provide the correct answer. However, they are unnecessarily complicated for many types of motion, such as pure rotation.

Rotating frame analysis is really important for cases where objects are **not pinned** to each other. When do you use rotating frames? When one object is sliding against another object, or two objects are not even in contact, but you want to know something about the motion relationship between them and/or are given information about the motion relationship between them.

Reference Frames:

In most of the preceding material, you have worked with the fixed reference frame O_{xyz}, and a translating (but not rotating) reference frame attached to the rigid body at a point A, frame A_{xyz}.

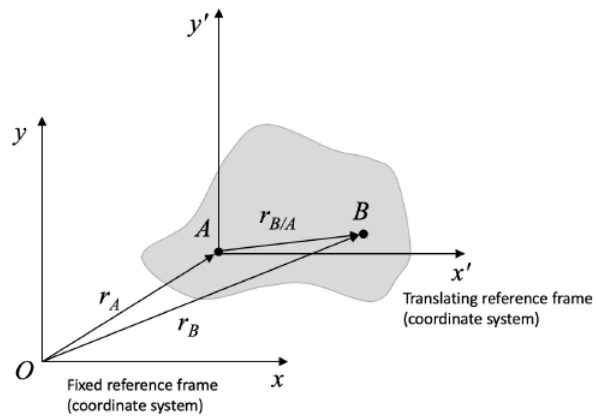

Figure 11.5.1: A rigid body with a translating (not rotating) reference frame $x'y'z'$ attached at point A, relative to the fixed frame xyz at O.

This has given us relative motion equations:

$$\text{Position:} \quad \vec{r}_{B/O} = \vec{r}_{A/O} + \vec{r}_{B/A} \tag{11.5.1}$$

$$\text{Velocity:} \quad \vec{v}_{B/O} = \vec{v}_{A/O} + \vec{v}_{B/A} = \vec{v}_{A/O} + \vec{\omega} \times \vec{r}_{B/A} \tag{11.5.2}$$

$$\text{Acceleration:} \quad \vec{a}_{B/O} = \vec{a}_{A/O} + \vec{a}_{B/A} = \vec{a}_{A/O} + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \tag{11.5.3}$$

Note that the notation $r_{B/O}$, where O is the fixed frame, can also be written as r_B. Both indicate an absolute position (or velocity or acceleration) - that is, a value with respect to the fixed frame O.

Rotating Frames:

Now, we will consider a reference frame that is attached to a point on the rigid body **and both rotating and translating with the rigid body**. We will introduce some extra terms to account for the rotation of the frame.

Consider a rigid body with frame $x'y'z'$ at point A. $x'y'z'$ moves and rotates with the body. A bug, B, is crawling along the body.

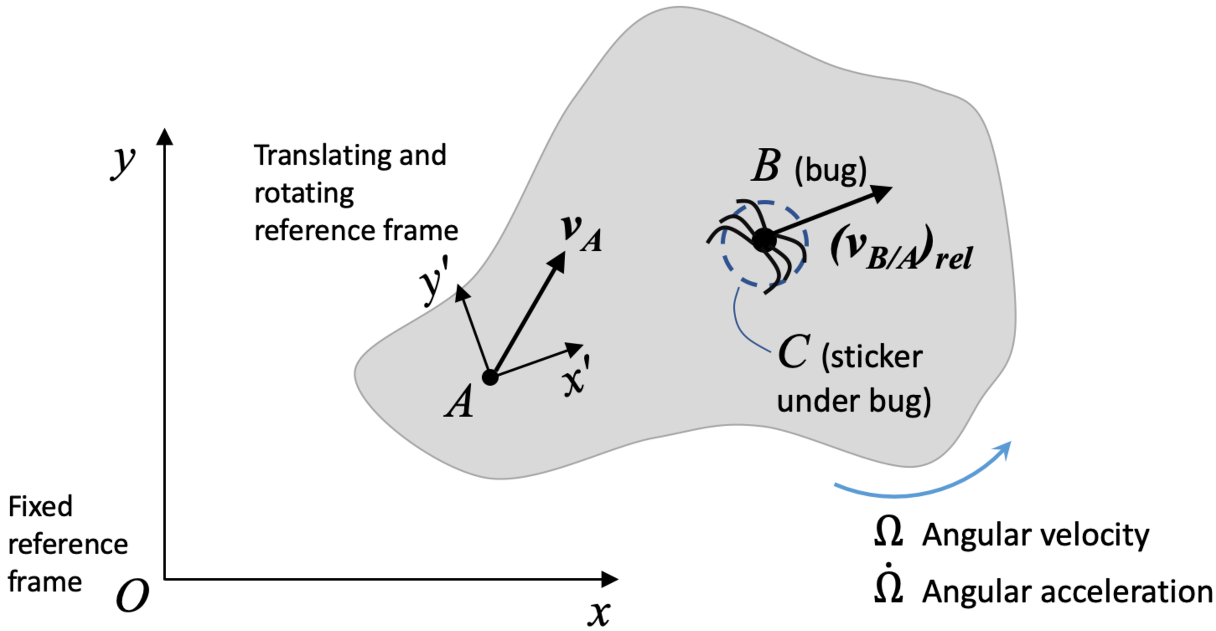

Figure 11.5.2: A rigid body with a translating AND rotating reference frame, $x'y'z'$, attached at point A. A bug, B, is crawling along the rigid body, with a velocity relative to the rotating frame. The bug is currently above a sticker, C, that is rigidly fixed to the rigid body.

For an observer sitting at A, the bug appears to be crawling away in a straight line. But for an observer sitting at O, the bug does not look like it is moving in a straight line, because the body it is crawling on is also moving (translating and rotating). We would like to describe the motion of the bug as seen by an observer at O. To remind ourselves that we are dealing with a rotating frame attached to one body, we use capital omega Ω) to denote the angular velocity of the object that the rotating frame is attached to, and $\dot{\Omega}$ for the angular acceleration of the object with the rotating frame attached. Finally, we use brackets and the subscript rel to denote values of velocity or acceleration expressed with respect to the rotating frame.

Recall that vectors have both magnitude and direction. We can express a vector in components with respect to the $\hat{i}$ and $\hat{j}$ unit vectors:

$$\vec{r}_{B/O} = r_{B/O,x}\hat{i} + r_{B/O,y}\hat{j} \tag{11.5.4}$$

Normally, when we take a time derivative of such an expression, the length of the vector (i.e. the $r_{B/O,x}$ and $r_{B/O,y}$ terms) change, but the unit vectors do not. In the case of the rotating frame, the unit vectors change. Now, they don't change in length - they remain unit vectors. But they are attached to a rotating body, which means they also rotate. That is, their directions change, and we have to account for that when deriving velocity and acceleration expressions.

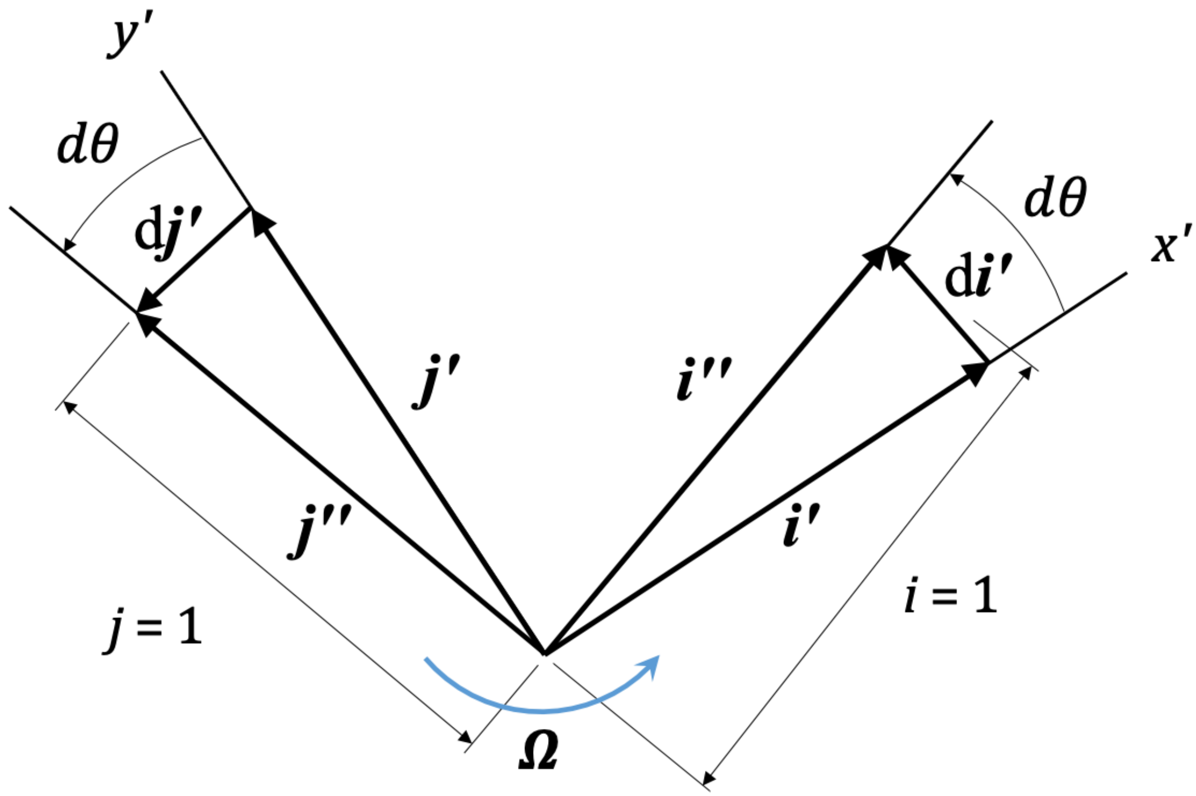

Figure 11.5.3: When the reference frame rotates with the rotating rigid body, the unit vectors change with time.

You can see we get small vectors $d\hat{i}'$ and $d\hat{j}'$ that describe the change in direction of the unit vectors, and the magnitude of that change is directly related to Ω, the angular velocity of the rigid body and the rotating frame.

$$\frac{d\hat{i}'}{dt} = \Omega\hat{k} \times \hat{i}' = \Omega\hat{j}' \tag{11.5.5}$$

$$\frac{d\hat{j}'}{dt} = \Omega\hat{k} \times \hat{j}' = -\Omega\hat{i}' \tag{11.5.6}$$

When we differentiate the relative position of the bug, B, with respect to the rotating frame, A, we have:

$$\frac{d\vec{r}_{B/A}}{dt} = \left(\dot{r}_{B/A,\,x'}\;\hat{i}' + \dot{r}_{B/A,\,y'}\;\hat{j}'\right) + \vec{\Omega} \times \left(r_{B/A,\,x'}\;\hat{i}' + r_{B/A,\,y'}\hat{j}'\right) \tag{11.5.7}$$

$$\text{New term:}\quad \left(\dot{r}_{B/A,\,x'}\;\hat{i}' + \dot{r}_{B/A,\,y'}\;\hat{j}'\right) = \left(\vec{v}_{B/A}\right)_{rel} \tag{11.5.8}$$

$$\text{Same as before:}\quad \vec{\Omega} \times \left(r_{B/A,\,x'}\;\hat{i}' + r_{B/A,\,y'}\;\hat{j}'\right) = \vec{\Omega} \times \vec{r}_{B/A} \tag{11.5.9}$$

The new term above describes the motion of the bug due to the rotation of the rigid body (and the $x'y'z'$ frame), while the familiar term above describes the motion of the bug as seen by an observer fixed to $x'y'z'$. Our final expression for the velocity of the bug with respect to the fixed frame is:

$$\vec{v}_{B/O} = \vec{v}_{A/O} + \vec{\Omega} \times \vec{r}_{B/A} + (\vec{v}_{B/A})_{rel} \tag{11.5.10}$$

This final expression has the new term, $(\vec{v}_{B/A})_{rel}$, which describes the motion of the bug as viewed from the rotating frame. You can note that if the bug is not moving, $(\vec{v}_{B/A})_{rel} = 0$, and we get the familiar expression from the translating, non-rotating frames. Thus, this expression is the most general relative velocity expression.

We can similarly derive an acceleration equation by differentiating the above velocity equation. The final expression is:

$$\vec{a}_{B/O} = \vec{a}_{A/O} + \dot{\vec{\Omega}} \times \vec{r}_{B/A} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{B/A}) + 2\vec{\Omega} \times (\vec{v}_{B/A})_{rel} + (\vec{a}_{B/A})_{rel} \tag{11.5.11}$$

Table 11.5.1: Physical correspondences of Equation 11.5.11 components

Motion of rotating frame attached to rigid body:	$\vec{a}_{A/O} + \dot{\vec{\Omega}} \times \vec{r}_{B/A} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{B/A})$
Coriolis acceleration - interaction of object motion with respect to rotating frame and motion of rotating frame:	$2\vec{\Omega} \times (\vec{v}_{B/A})_{rel}$
Motion of object with respect to rotating frame:	$(\vec{a}_{B/A})_{rel}$

Note that, for planar (2D) motion:

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{B/A}) = -\Omega^2 \vec{r}_{B/A} \tag{11.5.12}$$

Again, if the bug is sitting still, both rel terms become zero (it is not moving with respect to the rigid body), and we are left with the same relative motion expression used with translating frames. So, Equation 11.5.11 is the most general form of the relative acceleration equation and can always be used.

When tackling these problems, there are several important things to remember:

1. You need to pick the correct body to attach the rotating frame to. The correct body will have another body sliding against it. If the object of interest is pinned to the body you chose, it might not be correct.
2. It is often helpful to align your rotating frame such that you end up with a simplified "rel" term or terms. For example, line up the x' axis with the direction of movement of the bug, so that $(\vec{v}_{B/A})_{rel}$ is in the x'-direction.
3. The components of all terms of any one equation must be computed along the same $\hat{\imath}$ and $\hat{\jmath}$ directions. If the frames are not aligned, you must pick ONE FRAME and express all terms along that the directions of the ONE FRAME. (You may be able to switch between frames in different equations, but make sure you are careful with notation!)
4. When you are given information in a problem, pay careful attention to what frame is being referred to.
5. Make sure you include all the terms in the acceleration equation! It's easy to miss one. You should have five terms.

These are some of the most complex problems in planar kinematics, so take your time!

Exercise 11.5.1

A camera drone, D, flies over a car race in a curved trajectory (center O) with a constant ground-speed velocity of $v_D = 9\ m/s$. At the moment shown, car C is traveling with velocity of $v_c = 12\ m/s$ and an acceleration of $a_c = 2\ m/s^2$ as shown. Assume $d_1 = 7.5\ m,\ d_2$ = \ 3 m\).

- Find the velocity of the car <u>as observed by the camera on drone D</u> at this instant.
- Find the acceleration of the car <u>as observed by the camera on drone D</u> at this instant.

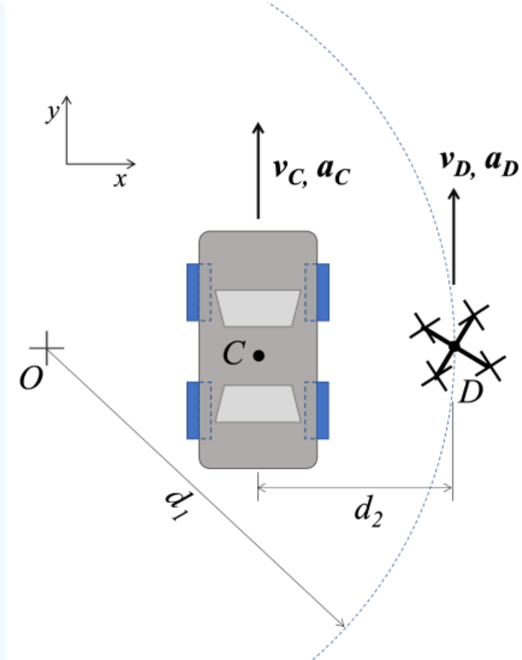

Figure 11.5.4: problem diagram for Example 11.5.1. A drone tracing a circular path around point O is currently straight to the right of a car, with the velocity and acceleration vectors of the two vehicles pointing in the same direction.

Solution

Video 11.5.1: Worked solution to example problem 11.5.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/nqs7bLBVm3g.

Example 11.5.2

A breathing exercise video graphic (somewhat similar to this one) shows a small circle moving in a constrained circular path (constant radius 80 cm) at a **constant** angular velocity of 0.4 rad/s around an expanding and contracting inner circle. The inner circle expands and contracts sinusoidally, from a minimum radius of 30 cm to a maximum radius of 60 cm. The distance from the center of the inner circle to a point on the edge of the inner circle can be described by the equation $r = 0.45 - 0.15\sin(\theta)$, where r is in meters and θ is the position of the small circle (zero at the x-axis).

Find the velocity and acceleration of point B on the edge of the inner circle as viewed by an observer on the small circle at point A (see part C of the figure below). $\theta = 45°$, $\alpha = 45°$

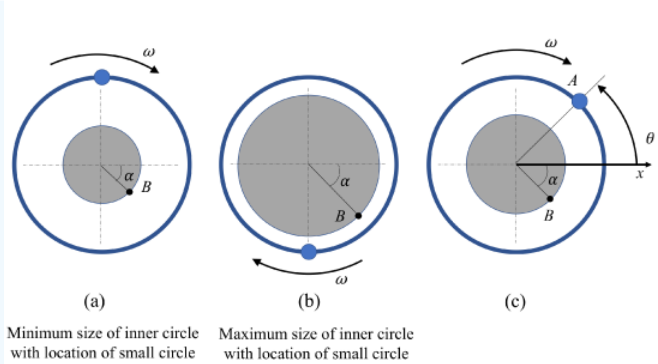

Figure 11.5.5: problem diagram for Example 11.5.2. A large circular outline rotates at a constant rate, changing the location of a small circle attached to it, while an inner circle concentric with the outline expands and contracts in radius.

Solution

Video 11.5.2: Worked solution to example problem 11.5.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/mL0PVJNnOBE.

11.6: Chapter 11 Homework Problems

Exercise 11.6.1

You are designing a bench grinder with an operating speed of 3600 rpm.

- If you want the grinder to reach its full operating speed in 4 seconds, what must the rate of angular acceleration be in radians per second squared?
- If the grinding wheel has a diameter of 8 inches, what will the speed of the surface of the wheel be?

Figure 11.6.1: A bench grinder.

Solution

$\alpha = 94.25 \,\frac{rad}{s}$

$v = 125.67 \, ft/s$

Exercise 11.6.2

A belt-driven system has an input at pulley A, which drives pulley B, which is attached with a solid shaft to pulley C, which drives pulley D. If the input is rotating at 60 rad/s counterclockwise, determine the angular velocity and direction of rotation for the output at D.

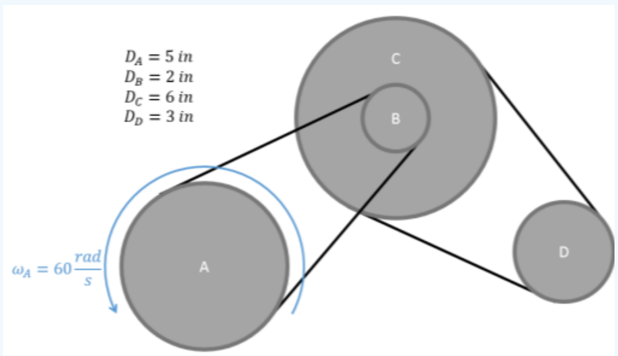

Figure 11.6.2: problem diagram for Exercise 11.6.2. A four-pulley system in which A and B are attached by a belt, C and D are attached by another belt, and B and C are mounted on the same shaft.

Solution

$\omega_D = 300 \,\frac{rad}{s}$ counterclockwise

Exercise 11.6.3

The piston in a piston and crank mechanism has the velocity and acceleration shown below. Using absolute motion analysis, determine the current angular velocity and angular acceleration for the crank.

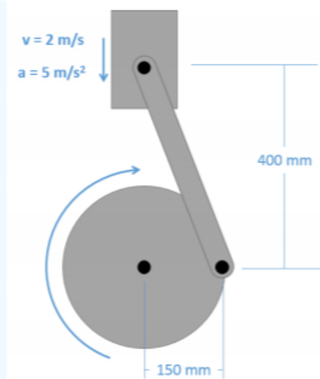

Figure 11.6.3: problem diagram for Exercise 11.6.3. A piston descends, causing the crank mounted on a fixed axle below it to rotate due to the bar that connects the piston and the crank.

Solution

$\omega = 13.33 \frac{rad}{s}$ clockwise

$\alpha = 100.16 \frac{rad}{s^2}$ clockwise

Exercise 11.6.4

A trapdoor is being opened with a hydraulic cylinder extending at constant rate of 0.7 m/s. If the door is currently at a twenty-degree angle as shown below, what is the current angular velocity and angular acceleration for the door?

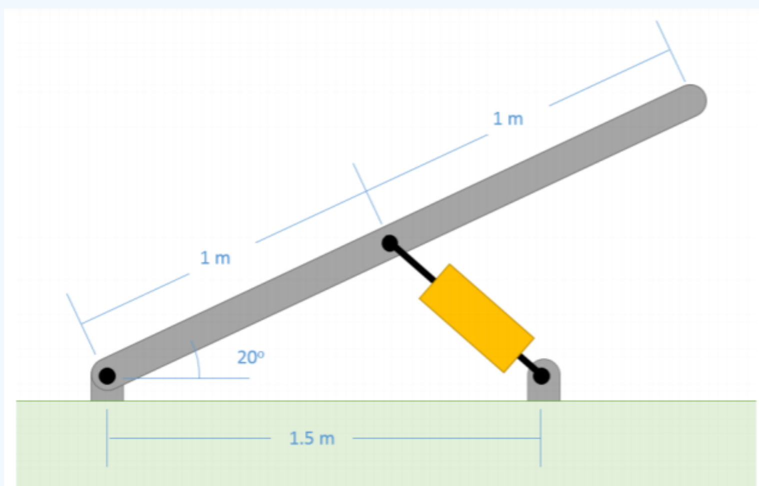

Figure 11.6.4: problem diagram for Exercise 11.6.4. Side view of a trapdoor being raised by the extension of a hydraulic cylinder that has one end attached to the door and the other fixed to a point on the ground.

Solution

$\dot{\theta} = 0.896 \frac{rad}{s}$, $\ddot{\theta} = -1.246 \frac{rad}{s^2}$

Exercise 11.6.5

A robotic arm experiences the angular velocities and accelerations shown below. Based on this information, determine the velocity and the acceleration of the end of the arm in the x and y directions.

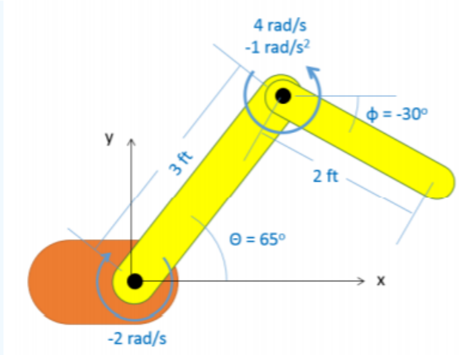

Figure 11.6.5: problem diagram for Exercise 11.6.5. A two-member robotic arm attached to a fixed base experiences rotation from two motors located at the joints.

Solution

$v_x = 9.44\ ft/s$, $v_y = 4.39\ ft/s$

$a_x = -33.78\ ft/s^2$, $a_y = 3.39\ ft/s^2$

CHAPTER OVERVIEW

12: Newton's Second Law for Rigid Bodies

12.1: Rigid Body Translation

With rigid bodies, we have to examine **moments** and at least the **possibility of rotation**, along with the forces and accelerations we examined with particles. Some rigid bodies will translate but not rotate (translational systems), some will rotate but not translate (fixed axis rotation), and some will rotate and translate (general planar motion). Overall, we will start with an examination of translational systems, then examine fixed axis rotation, then pull everything together for general planar motion.

Figure 12.1.1: The braking car in this picture is an example of a translational system. Though the car experiences a significant deceleration, it does not experience any significant rotation while it slows down. Public domain image by Sgt. Amber Blanchard.

As the start of our analysis, we will go back to Newton's Second Law. Since this is a rigid body system, we include both the translational and rotational versions.

$$\sum \vec{F} = m * \vec{a} \tag{12.1.1}$$

$$\sum \vec{M} = I * \vec{\alpha} \tag{12.1.2}$$

As we did with particles, we can break the vector force equation into components, turning the one vector equation into two scalar equations (in the x and y directions respectively). As for the moment equation, a translational system by definition will have zero angular acceleration. With the angular acceleration being zero, the sum of the moments must all be equal to zero. This is similar to statics problems; however, there is one big difference we must take into account. **The moments must be taken about the center of mass of the body.** Setting the moments to zero about other points will lead to invalid solutions for any body experiencing an acceleration. Putting these specifics into action, we wind up with the three base **equations of motion** below. To solve for unknown forces or accelerations, we simply draw a free body diagram, put the knowns and unknowns into these equations, and solve for the unknowns.

$$\sum F_x = m * a_x \tag{12.1.3}$$

$$\sum F_y = m * a_y \tag{12.1.4}$$

$$\sum M_G = 0 \tag{12.1.5}$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/V4Gy-tyHupE.

Example 12.1.1

A refrigerator is 2.5 feet wide and 6 feet tall, and weighs 80 lbs. The center of mass is 1.25 feet from either side and 2 feet up from the base. If the refrigerator is on a conveyor belt that is accelerating the fridge at a rate of 1 ft/s^2, what are the normal forces at each of the feet?

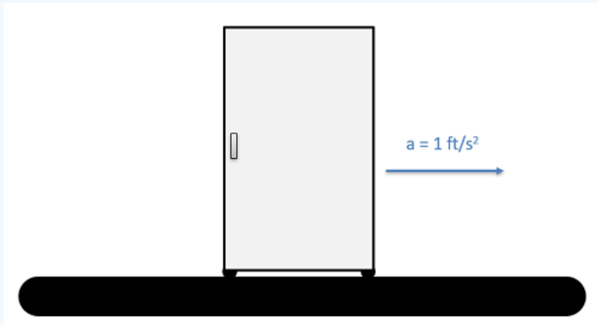

Figure 12.1.2: problem diagram for Example 12.1.1. A refridgerator of the dimensions described above is on a conveyor belt that accelerates the fridge to the right at a rate of 1 ft/s^2.

Solution

Video 12.1.2: Worked solution to example problem 12.1.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/59GRMn_ZvJk.

12.2: Fixed-Axis Rotation

With rigid bodies, we have to examine **moments** and at least the **possibility of rotation** along with the forces and accelerations we examined with particles. Some rigid bodies will translate but not rotate (translational systems), some will rotate but not translate (fixed axis rotation), and some will rotate and translate (general planar motion). Here we will examine rigid body rotation about a fixed axis. As the name would suggest, fixed axis rotation is the analysis of any rigid body that rotates about some axis that does not move. Many devices rotate about their center, though the objects do not need to rotate about their center point for this analysis to work.

Figure 12.2.1: The wheel on this pitching machine is an example of fixed axis rotation with the axis of rotation at the center of mass. Image by Abigor, license CC-BY 2.0.

We will again start with Newton's Second Law. Since this is a rigid body system, we include both the translational and rotational versions.

$$\sum \vec{F} = m * \vec{a} \tag{12.2.1}$$

$$\sum \vec{M} = I * \vec{\alpha} \tag{12.2.2}$$

By setting up free body diagrams, determining the equations of motion using Newton's Second Law, and solving for the unknowns, we can find forces based on the accelerations or vice versa.

Balanced Rotation

If the center of mass of the body is at the axis of rotation, which is known as balanced rotation, then acceleration at that point will be equal to zero. The pitching machine above is an example of a balanced rotation, and most fixed axis systems will be intentionally built to be balanced. With the acceleration of the center of mass being zero, the sum of the forces in both the x and y directions must be also be equal to zero.

$$\sum F_x = 0 \tag{12.2.3}$$

$$\sum F_y = 0 \tag{12.2.4}$$

In addition to the force equations, we will can also use the moment equations to solve for unknowns. In simple planar motion, this will be a single moment equation which we take about the axis of rotation or center of mass (remember that they are the same point in balanced rotation).

$$\sum M_O = I_O * \alpha \tag{12.2.5}$$

Unbalanced Rotation

When the center of mass is not located on the axis of rotation, the center of mass will be accelerating and therefore forces will be exerted to cause that acceleration. In perfectly anchored systems these will be forces exerted by the bearings, though these forces can often be felt as vibrations in real systems.

Figure 12.2.2: Many video game controllers use motors to rotate the small masses where the center of mass is not at the axis of rotation. This unbalanced rotation results in forces (felt as vibrations) needed to accelerate the center of mass of the spinning masses. Image by unknown author under a CC0 license.

The kinematics equations discussed in the previous chapter can be used to determine the acceleration of a point on a rotating body, that point being the center of mass in this case. After determining those accelerations, they can be put into force equations, most likely using the r and θ directions.

$$\sum F_r = ma_r \tag{12.2.6}$$

$$\sum F_\theta = ma_\theta \tag{12.2.7}$$

Note that as the body rotates, the direction of the acceleration and the direction of the forces change. Also note that the further the center of mass is from the axis of rotation, the larger the mass. The larger the angular velocity, the larger these forces will be.

To supplement the force equations, we can use a moment equation about either the axis of rotation or the center of mass, as these are no longer the same point. Whichever is chosen, just be sure to be consistent in taking the moments and the mass moment of inertia about the same point.

$$\sum M_O = I_O * \alpha \qquad \text{or} \qquad \sum M_G = I_G * \alpha \tag{12.2.8}$$

More information on how to calculate the mass moment of inertia for a body can be found in Appendix 2.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/EGmfwhYbRew.

Example 12.2.1

A hard drive platter 8 cm in diameter accelerates at a constant rate of 150 rad/s^2. If the hard drive weighs a uniformly distributed 0.05 kg and we approximate the hard drive as a flat circular disc, what moment does the motor need to exert to accelerate the drive at this rate?

Figure 12.2.3: A picture of a hard drive that emphasizes the flat circular platter.

Solution

Video 12.2.2: Worked solution to example problem 12.2.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/grJ8WYsGitw.

Example 12.2.2

The drum in a washing machine can be approximated as a cylinder 0.4 meters in diameter and 0.3 meters in height with a uniformly distributed mass of 35 kilograms when full. If we wish to achieve an acceleration of 15 rad/s^2, what torque must the motor exert at the center of the drum?

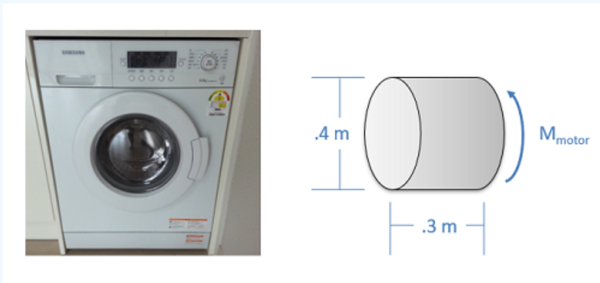

Figure 12.2.4: problem diagram for Example 12.2.2. A front-loading washing machine, and an approximation of the machine's drum as a cylinder rotated about its axis by the motor.

Solution

Video 12.2.3: Worked solution to example problem 12.2.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/7G7HnwsyLXk.

12.3: Rigid-Body General Planar Motion

In **general planar motion**, bodies are both rotating and translating at the same time. As a result, we will need to relate the forces to the acceleration of the center of mass of the body as well as relating the moments to the angular accelerations.

Figure 12.3.1: This tire being rolled along the ground is an example of general planar motion. The tire is both translating and rotating as it is pushed along. Image by Joy Agyepong, CC-BY-SA 4.0.

To analyze a body undergoing general planar motion, we will start by drawing a free body diagram of the body in motion. Be sure to identify the center of mass, as well as identifying all known and unknown forces, and known and unknown moments acting on the body. It is also sometimes helpful to label any key dimensions as well as using dashed lines to identify any known accelerations or angular accelerations.

Next, we move on to identifying the equations of motion. At its core, this means going back to Newton's Second Law. Since this is a rigid body system, we include both the translational and rotational versions.

$$\sum \vec{F} = m * \vec{a} \tag{12.3.1}$$

$$\sum \vec{M} = I * \vec{\alpha} \tag{12.3.2}$$

As we did with the previous translational systems, we will break the force equation into components, turning the one vector equation into two scalar equations. Additionally, it's important to always use the center of mass for the accelerations in our force equations and take the moments and moment of inertia about the center of mass for our moment equation.

$$\sum F_x = m * a_x \tag{12.3.3}$$

$$\sum F_y = m * a_y \tag{12.3.4}$$

$$\sum M_{COM} = I * \alpha \tag{12.3.5}$$

Plugging the known forces, moments, and accelerations into the above equations, we can solve for up to three unknowns. If more than three unknowns exist in the equations, we will sometimes have to go back to kinematics to relate quantities such as acceleration and angular acceleration.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/jhVDM7Zw_LM.

Example 12.3.1

A cylinder with a radius of 0.15 m and a mass of 10 kg is placed on a ramp at a 20-degree angle. If the cylinder is released from rest and rolls without slipping, what is the initial angular acceleration of the cylinder and the time required for the cylinder to roll 5 meters?

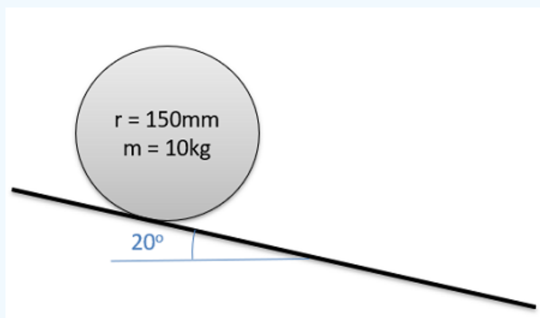

Figure 12.3.2: problem diagram for Example 12.3.1. A uniform cylinder rolls down a ramp that has a 20° incline.

Solution

Video 12.3.2: Worked solution to example problem 12.3.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/oQFVVC3SzZ0.

Example 12.3.2

The cable spool shown below has a weight of 50 lbs and has a moment of inertia of 0.28 slug-ft^2. Assume the spool rolls without slipping when we apply a 50-lb tension in the cable.

- What is the friction force between the spool and the ground?
- What is the acceleration of the center of mass of the spool?

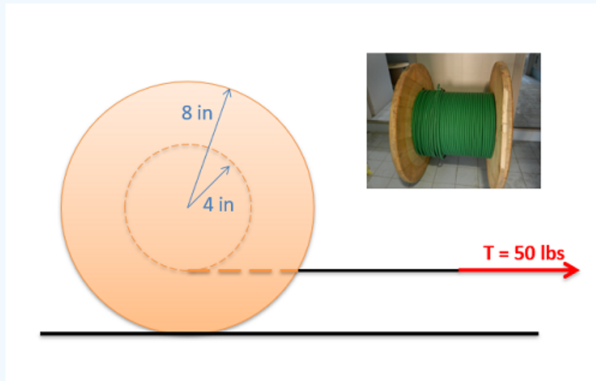

Figure 12.3.3: problem diagram for Example 12.3.2. A cable spool, consisting of a central cylinder the cable wraps around sandwiched between two larger disks at its bases, is pulled to the right by a tension force on the free end of the cable. Spool image by Seeweb, CC-BY-SA 2.0.

Solution

Video 12.3.3: Worked solution to example problem 12.3.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/RAhLP-kMYaQ.

Example 12.3.3

You are designing a Frisbee launcher to launch a 40-cm-diameter, 0.6 kg Frisbee that can be modeled as flat circular disc. If you want the Frisbee to have a linear acceleration of 20 m/s^2 and an angular acceleration of 50 rad/s^2 as shown below, what should F_1 and F_2 be?

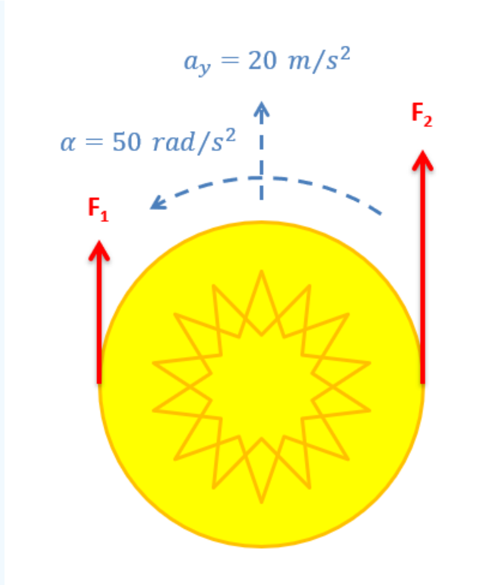

Figure 12.3.4: problem diagram for Example 12.3.3. A Frisbee, represented as a circle, accelerates both linearly and angularly as a result of experiencing two forces applied in the same direction at locations on opposite sides of a diameter.

Solution

Video 12.3.4: Worked solution to example problem 12.3.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/DY9B7oU0j7k.

Example 12.3.4

A pickup truck is carrying a 30-kg, 6-meter-long ladder at a 35-degree angle as shown below. The ladder is wedged against the tailgate at A and makes contact with the roof of the truck at B. The distance from A to B is 2 meters. At what rate of acceleration would we expect the ladder to start to rotate upwards?

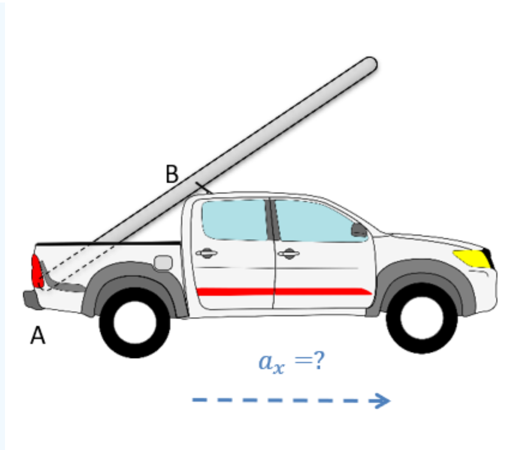

Figure 12.3.5: problem diagram for Example 12.3.4. A moving pickup truck carries a ladder that leans forward, with its lower end propped against the tailgate.

Solution

Video 12.3.5: Worked solution to example problem 12.3.4, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/UO5XsDFxQoY.

12.4: Multi-Body General Planar Motion

In cases where multiple connected rigid bodies are undergoing some sort of motion, we can extend our analysis of general planar motion to this multi-body situation. In these cases, which we will call multi-body kinetics problems, we will analyze each body independently as we did for general planar motion, but we will also need to pay attention to the Newton's Third Law pairs. Each body will have forces exerted on it by the surrounding bodies, and it will exert equal and opposite forces back through those same connections.

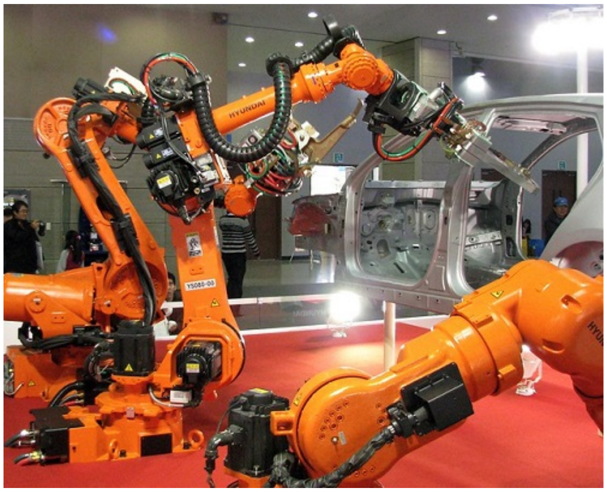

Figure 12.4.1: These robotic arms are a good example of a multi-body kinetics problem. As one section of the arm accelerates, it will exert forces on the other arm sections it is connected to. Image by Chris Chesher, CC-BY-NC-SA 2.0.

To analyze a multi-body system, we will start by drawing a free body diagram of each body in motion. Be sure to identify the center of mass, as well as identifying all known and unknown forces, and known and unknown moments acting on the body. When drawing forces at connection points, be sure to make the forces equal and opposite on the connected body to satisfy Newton's Third Law. It is also sometimes helpful to label any key dimensions as well as using dashed lines to identify any known accelerations or angular accelerations. Often, you will need to solve a kinematics problem using absolute motion analysis or relative motion analysis in order to determine the accelerations of the centers of mass and the angular accelerations for each body. Make sure all these accelerations are with respect to ground.

Next we move onto identifying the equations of motion for each body in the system. In two dimensions, we will use the same three equations we used for general planar motion. Be sure to find the the accelerations of all the centers of masses, find all moments about the center of mass, and take the mass moments of inertia about the center of mass of each body.

$$\sum F_x = m * a_x \tag{12.4.1}$$

$$\sum F_y = m * a_y \tag{12.4.2}$$

$$\sum M_G = I_G * \alpha \tag{12.4.3}$$

Plugging the known forces, moments, and accelerations into the above equations we can solve for up to three unknowns per body. If more than three unknowns exist in any one set of equations, you will need to start with an adjacent body. Once unknown forces are determined on one body, they can become knowns on the connected body, reducing the number of unknowns to solve for.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/-tAP08srNrU.

Example 12.4.1

A robotic arm has two sections (OA and AB), with section OA having a mass of 10 kg and section AB having a mass of 7 kg. Treat each section as a slender rod. If we wish to accelerate member AB from a standstill at a rate of 3 rad/s² and keep the left section stationary, what moments must we exert at joints O and A?

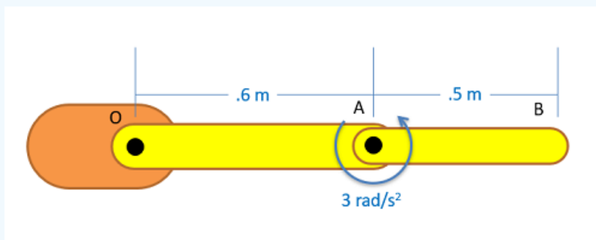

Figure 12.4.2: problem diagram for Example 12.4.1. A robotic arm consists of two segments that are currently horizontal, with the left end of the left segment being attached to a base and the right segment undergoing angular acceleration.

Solution

Video 12.4.2: Worked solution to example problem 12.4.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/zR_uhVM1uH0.

12.5: Chapter 12 Homework Problems

Exercise 12.5.1

The SUV shown below has an initial velocity of 90 ft/s. It slams on its brakes, coming to a stop over a distance of 300 feet. If the car has a weight of 3500 lbs and a center of mass as shown below, what are the normal forces at the front wheels? What are the normal forces at the back wheels?

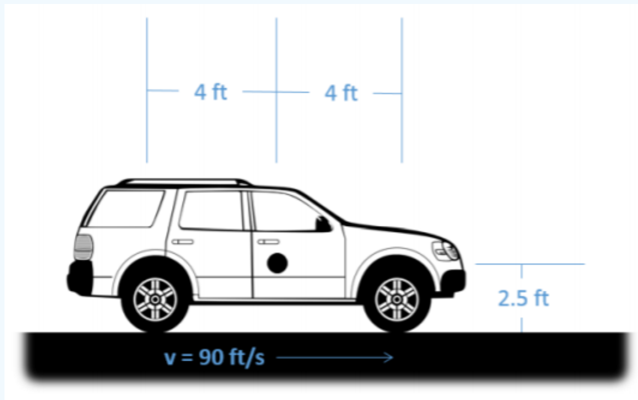

Figure 12.5.1: problem diagram for Exercise 12.5.1. A car traveling in a straight line applies its brakes, coming to a gradual stop.

Solution

$$F_{N_{rear}} = 1291.4\ lbs$$

$$F_{N_{front}} = 2208.6\ lbs$$

Exercise 12.5.2

A ring-shaped space station can be approximated as a thin ring 60 meters in diameter with a mass of 500,000 kg. The space station has a set of thrusters able to exert equal and opposite forces as shown below. If we want to cause an angular acceleration of 0.1 rad/s² in the space station, what is the force required from each thruster?

Figure 12.5.2: problem diagram for Exercise 12.5.2. A ring-shaped space station is given the specified angular acceleration through the firing of two thrusters located on either side of a diameter, pointing in opposite directions.

Solution

$$F_{thruster} = 750\ kN$$

Exercise 12.5.3

A 50-kg barrel with a diameter of 0.75 meters is placed on a 20-degree slope. Assuming the barrel rolls without slipping, what will the acceleration of the barrel's center of mass be?

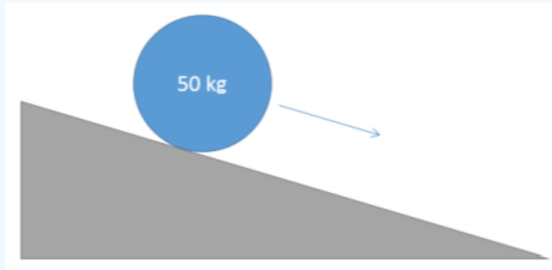

Figure 12.5.3: problem diagram for Exercise 12.5.3. A 50-kg barrel with a 0.75-meter diameter rolls down a 20° incline without slipping.

Solution

$a_x = 2.24\ m/s^2$

Exercise 12.5.4

A 3-meter-long, 25-kg beam is supported by two cables as shown below. You can treat the beam as a slender rod. Assume that we want the left end of the beam at point A to remain at a constant height while the right end of the beam at point B accelerates upwards at a rate of 1 m/s².

- What is the rate of acceleration of the center of the beam and the rate of angular acceleration for the beam?
- What will T_1 and T_2 need to be to achieve these accelerations?

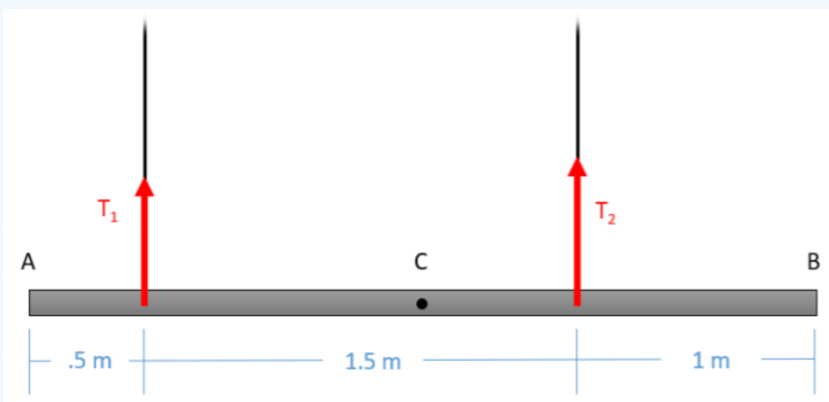

Figure 12.5.4: problem diagram for Exercise 12.5.4. A horizontal beam is held in the air by the tension forces from two vertical cables attached near the beam's ends.

Solution

$a_{C\backslash \mathrm{y}} = 0.5\ m/s^2,\ \alpha = 0.333\ \frac{rad}{s}$

$T_1 = 81.75\ N,\ T_2 = 176\ N$

Exercise 12.5.5

You are modeling the robotic arm shown below. Treat each section of the arm as a slender rod. Section OA weighs 30 lbs and section AB weighs 18 lbs. If we want the <u>relative</u> angular accelerations and velocities shown below, what should the motor torques be at O and A? (This is a top-down view of the robot arm.)

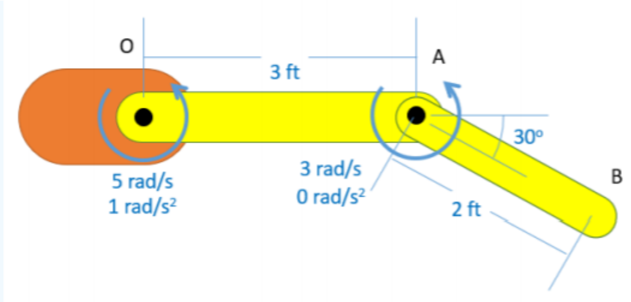

Figure 12.5.5: problem diagram for Exercise 12.5.5. Top-down view of a two-segment robotic arm with one end attached to a fixed base, with motors at the two joints providing rotation.

Solution

$M_O = -3.9 \; ft\text{-}lbs$

$M_A = -19.3 \; ft\text{-}lbs$

CHAPTER OVERVIEW

13: Work and Energy in Rigid Bodies

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13.1: Conservation of Energy for Rigid Bodies

The concepts of **Work** and **Energy** provide the basis for solving a variety of kinetics problems. Generally, this method is called the **Energy Method** or the **Conservation of Energy**, and it can be boiled down to the idea that the work done to a body will be equal to the change in energy of that body. Dividing energy into kinetic and potential energy pieces as we often do in dynamics problems, we arrive at the following base equation for the conservation of energy.

$$W = \Delta KE + \Delta PE \tag{13.1.1}$$

It is important to notice that unlike Newton's Second Law, the above equation is **not** a vector equation. It does not need to be broken down into components which can simplify the process. However, we only have a single equation and therefore can only solve for a single unknown, which can limit this method.

Work in Rigid Body Problems:

For work done to a rigid body, we must consider any force applied over a distance as we did for particles, as well as any moment exerted over some angle of rotation. If these are **constant** forces and **constant** moments, we simply multiple the force times the distance and the moment times the angle of rotation to find the overall work done in the problem. As with particles, these are the components of forces in the direction of travel, with forces opposing the motion counting as negative work. Similarly, these are moments in the direction of rotation, with moments opposing the rotation counting as negative work. Both types of work are additive, with all work being lumped together for analysis.

$$W = F * d + M * \Delta\theta \tag{13.1.2}$$

In instances of **non-constant** forces and **non-constant** moments, we will need to integrate the forces and moments over the distance traveled and the angle of rotation, respectively.

$$W = \int_{x_1}^{x_2} F(x)\,dx + \int_{\theta_1}^{\theta_2} M(\theta)\,d\theta \tag{13.1.3}$$

Energy:

In rigid bodies, as with particles, we will break energy into **kinetic energy** and **potential energy**. Kinetic energy is the the energy mass in motion, while potential energy represents the energy that is stored up due to the position or stresses in a body.

In its equation form, the kinetic energy of a rigid body is represented by one-half of the mass of the body times its velocity squared, plus one-half of the mass moment of inertia times the angular velocity squared. If we wish to determine the change in kinetic energy, we would simply take the final kinetic energy minus the initial kinetic energy.

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{13.1.4}$$

$$\Delta KE = \left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) + \left(\frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2\right) \tag{13.1.5}$$

Potential energy, unlike kinetic energy, is not really energy at all. Instead, it represents the work that a given force will potentially do between two instants in time. Potential energy can come in many forms, but the two we will discuss here are gravitational potential energy and elastic potential energy. These represent the work that the gravitational force and a spring force will do, respectively. We often use these potential energy terms in place of the work done by gravity or springs respectively. When including these potential energy terms, it's important to not also include the work done by gravity or spring forces.

The change in gravitational potential energy for any system is represented by the mass of the body, times the value g (9.81 m/s^2 or 32.2 ft/s^2 on the earth's surface), times the vertical change in height between the start position and the end position. In equation form, this is as follows.

$$\Delta PE = m * g * \Delta h \tag{13.1.6}$$

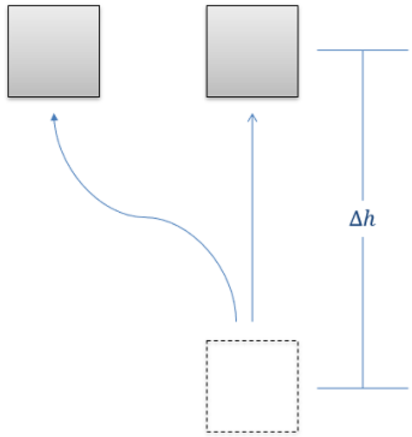


Figure 13.1.1: When finding the change in gravitational potential energy, we multiply the object's mass by g (giving us the weight of the object) and then multiply that by the change in the height of the object, regardless of the path taken.

Unlike work and kinetic energy, there is no rotational version of gravitational potential energy, so this is exactly the same as we had for particles.

To determine the change in elastic potential energy, we will have to identify any linear springs (as we had for particles), as well as any torsional springs and the spring constants for each of these springs.

Figure 13.1.2: The torsional spring in this mousetrap releases its elastic potential energy to slam the trap shut. Public domain image by Evan-Amos.

To find the change in elastic potential energy, we will need to know the stiffness of any spring in the problem (represented by k, in units of force per distance for linear springs or moment per angle of twist for torsional springs) as well as the distance or angle the spring has been stretched above or below its natural resting position. This difference from resting position is represented by distance x for linear springs, or angle θ for torsional springs. Once we have those values, the elastic potential energy can be calculated by multiplying one-half of the stiffness by the distance x squared or the angle θ squared. To find the change in elastic potential energy, we simply take the final elastic potential energy minus the initial elastic potential energy.

$$\Delta PE_{linear\ spring} = \frac{1}{2}k\, x_f^2 - \frac{1}{2}k\, x_i^2 \tag{13.1.7}$$

$$\Delta PE_{torsional\ spring} = \frac{1}{2}k\, \theta_f^2 - \frac{1}{2}k\, \theta_i^2 \tag{13.1.8}$$

Going back to our original conservation of energy equation, we simply plug the appropriate terms on each side (work on the left and energies on the right) and balance the two sides to solve for any unknowns. Terms that do not exist or do not change (such as elastic potential energy in a problem with no springs, or ΔKE in a problem where there is no change in the speed of the body) can be set to zero. Again, there is only one equation, so we can only solve for a single unknown unless we supplement the conservation of energy equation with other relationship equations.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/YVkzjzHSW-g.

Example 13.1.1

The turntable on a record player consists of a disk 12 inches in diameter with a weight of 5 lbs. The motor accelerates the turntable from rest to its operating speed of 33.33 rpm in one rotation. What is the work done by the motor? What is the average torque the motor exerted?

Figure 13.1.3: A turntable on a record player. Image by Ron Clausen, CC-BY-SA 4.0.

Solution

Video 13.1.2 Worked solution to example problem 13.1.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/UPHgU-o80Q0.

Example 13.1.2

A system as shown below is used to passively slow the lowering of a gate. The gate can be approximated as a flat plate on its edge with a mass of 25 kilograms and a height of 2 meters. Assume the spring is unstretched as shown in the diagram.

- What would the angular velocity of the gate be without the spring?
- If we want to reduce the angular velocity at the bottom to 25% of its original value, what should the spring constant be?

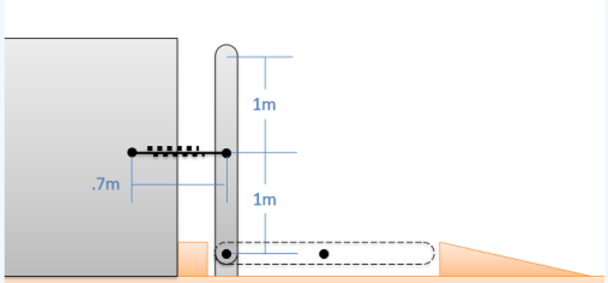


Figure 13.1.4: problem diagram for Example 13.1.2. Top-down view of a hinged gate in a currently open position, attached by a spring to a wall running parallel to it that will slow the gate's motion as it swings shut.

Solution

Video 13.1.3: Worked solution to example problem 13.1.2, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/HCAC0tNR18w.

Example 13.1.3

A 5-kilogram spherical ball with a radius of 0.05 meters is placed on a ramp as shown below. If the ball rolls without slipping, what is the velocity of the ball at the bottom of the ramp?

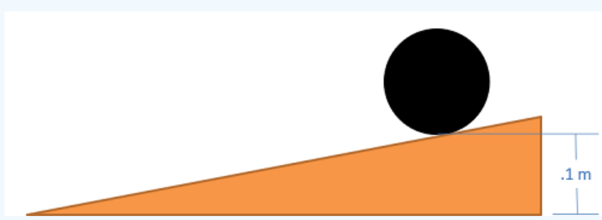


Figure 13.1.5: problem diagram for Example 13.1.3. A spherical ball is placed at the top of a ramp, 0.1 meters above the ground, and rolls without slipping.

Solution

Video 13.1.4: Worked solution to example problem 13.1.3, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/p2pvQ0kCWIU.

Example 13.1.4

A 16-kilogram half-cylinder is placed on a hard, flat surface as shown below and released from rest. What will the maximum angular velocity be as it rocks back and forth?

Figure 13.1.6: problem diagram for Example 13.1.4. A cylinder 50 cm in diameter is sliced in half lengthwise, then balanced on a flat, hard surface so that only one of its long, straight edges is in contact with said surface.

Solution

Video 13.1.5: Worked solution to example problem 13.1.4, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/HpRtDQjoEBA.

Example 13.1.5

A mechanism consists of two 3-kilogram wheels connected to a 2-kilogram bar as shown below. Based on the dimensions in the diagram, what is the minimum required initial velocity for the wheels to ensure the mechanism makes it all the way through one rotation without rocking backwards?

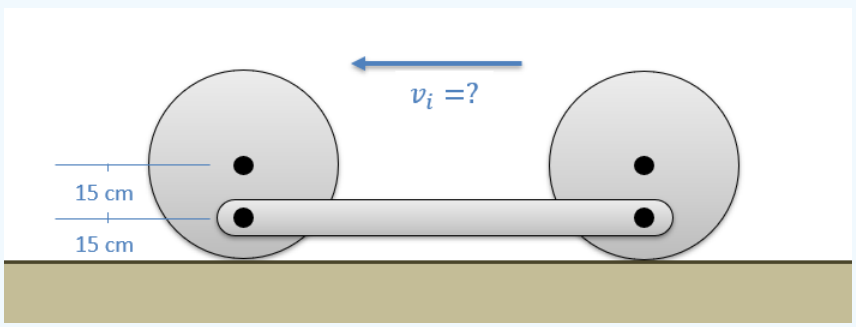


Figure 13.1.7: problem diagram for Example 13.1.5. A pair of wheels is connected by a bar of fixed length, constraining them to roll together.

Solution

Video 13.1.6: Worked solution to example problem 13.1.5, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/8cjQ1S6yOTc.

13.2: Power and Efficiency in Rigid Bodies

Related to the concepts of work and energy are the concepts of **power** and **efficiency**. At its core, power is the rate at which work is being done, and efficiency is the percentage of useful work or power that is transferred from input to output of some system.

Power

Power at any instant is defined as the derivative of work with respect to time.

$$P=\frac{dW}{dt} \tag{13.2.1}$$

If we look at the average power over a set period, we can simply measure the work done and divide that by the time. Work for a rigid body is defined as the force times the distance the center of mass travels, plus the moment times angle of rotation (in radians) for our body.

$$P_{ave}=\frac{W}{t}=\frac{F*d+M*\Delta\theta}{t} \tag{13.2.2}$$

Using the definition of velocity (distance over time) and the definition of angular velocity ($\Delta\theta$ over time), we arrive at a third equation for power at a given instant.

$$P=F*v+M*\omega \tag{13.2.3}$$

The common units of power are **watts** for the metric system, where one watt is defined as one Joule per second or one Newton-meter per second, and **horsepower** in the English system, where one horsepower is defined as 550 foot-pounds per second.

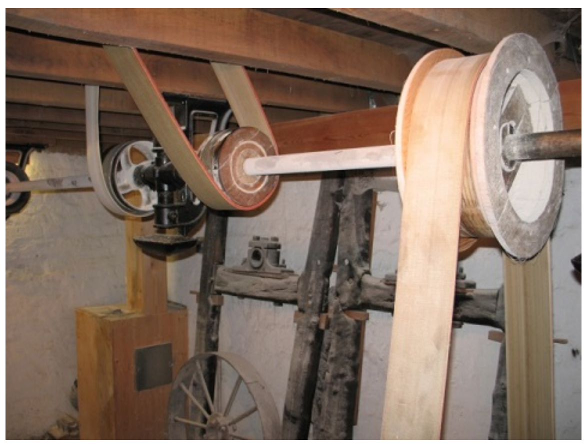


Figure 13.2.1: The drive shaft in this mill is used to transfer power from the input to the output. If we multiplied the toque in the shaft (in Newton-meters) by the angular velocity (in radians per second), we would get the power being transmitted in watts (where a watt is a Newton-meter per second). Image by Ian Petticrew, CC-BY-SA 2.0.

Efficiency

Any devices with work/power inputs and outputs will have some loss of work or power between that input and output due to things like friction. While energy is always conserved, some energies such as heat may not be considered useful. A measure of the useful work or power that makes it from the input of a device to the output is the efficiency. Specifically, efficiency is defined as the work gotten out of a device divided by the work put into the device. With power being the work over time, efficiency can also be described as power out of a device divided by the power put into a device (the time term would cancel out, leaving us with our original definition).

$$\eta=\frac{W_{out}}{W_{in}}=\frac{P_{out}}{P_{in}} \tag{13.2.4}$$

It is impossible to have efficiencies greater than one (or 100%) because that would be a violation of the conservation of energy; however, for most devices we wish to get the efficiencies as close to one as possible. This is not only because it does not waste work/power, but also because any work or power that is "lost" in the device will be turned into heat that may build up.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/krZ1w5RNdwg.

Example 13.2.1

The input to a gearbox has a measured 32 foot-pounds of torque at 700 rpm. The output has 207 foot-pounds of torque at 100 rpm.

- What is the power at the input?
- What is the power at the output?
- What is the efficiency of the gearbox?

Figure 13.2.2: A model of a gearbox.

Solution

Video 13.2.2: Worked solution to example problem 13.2.1, provided by Dr. Majid Chatsaz. YouTube source: https://youtu.be/Zauv-wZN_CE.

13.3: Chapter 13 Homework Problems

Exercise 13.3.1

An impact-testing device consists of a 20-kilogram box supported by two slender 5-kilogram rods. The two rods are set up in parallel so that the box remains level as it swings. If the whole system is released from the horizontal position shown below, what is the velocity of the box after it has traveled 90°?

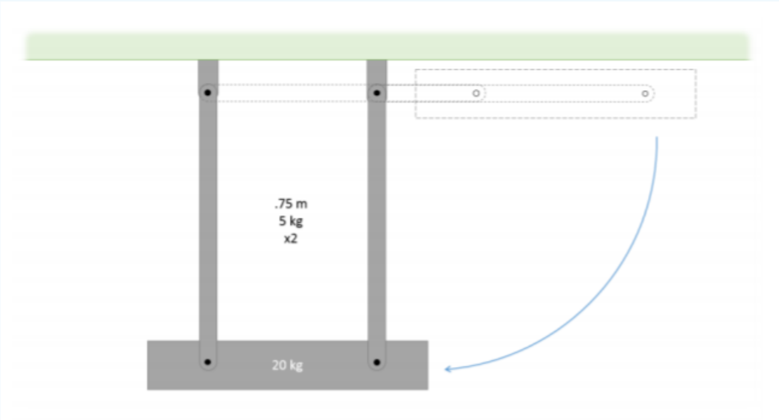


Figure 13.3.1: problem diagram for Exercise 13.3.1. An impact-testing device composed of a pair of rods swivel-mounted on a heavy box and attached to the ceiling, allowing for the box to be raised close to the ceiling and to swing downwards when released, remaining parallel throughout.

Solution

$$v = 3.97\ m/s$$

Example 13.3.2

A 40-lb door with a width of 36 inches is attached to a spring with an unstretched length of 4 inches, designed to close the door when left open. The spring is anchored as shown below when closed (solid outline is closed, dotted outline is open 90°). If we want the door to have an angular velocity of 0.2 rad/s upon closing when released from rest at the 90° open position, what should the spring constant of the spring be? (This is the top view of the door below)

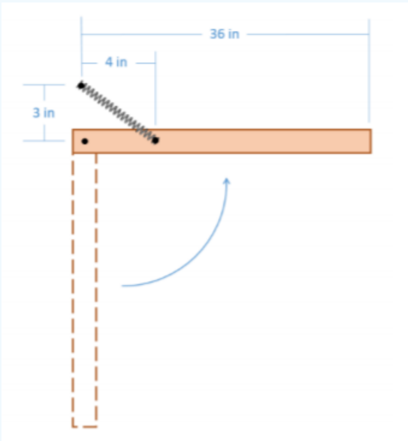


Figure 13.3.2: problem diagram for Exercise 13.3.2. Top-down view of the closed and open positions a door designed to close when left open, due to the action of a spring. The unstretched spring is shown attached to the door and the wall beside the hinge, in the closed position.

Solution

$$k = 2.68\ lb/ft = 0.224\ lb/in.$$

CHAPTER OVERVIEW

14: Impulse and Momentum in Rigid Bodies

14.1: Impulse-Momentum Equations for a Rigid Body

The concepts of **impulse** and **momentum** provide a third method of solving kinetics problems in dynamics. Generally this method is called the **Impulse-Momentum Method**, and it can be boiled down to the idea that the impulse exerted on a body over a given time will be equal to the change in that body's momentum. In a rigid body we will be concerned with not only linear impulse and momentum, but also **angular** impulse and momentum. The linear and angular impulse momentum equations are below, with the new term for angular impulse $(\vec{K})$ starting out the angular impulse momentum equation.

$$\vec{J}=m\vec{v}_f-m\vec{v}_i \tag{14.1.1}$$

$$\vec{K}=I_G\vec{\omega}_f-I_G\vec{\omega}_i \tag{14.1.2}$$

For two-dimensional problems, we can break the linear impulse equation down into two scalar components to solve. In the case of planar problems, we simply need to break all forces and velocities into x and y components - since all rotation will be about the z-axis, the angular impulse momentum equation remains a single equation. Notice, however, that we will need to take the mass moment of inertia about the center of mass of the body into account; similarly, we will use the velocity of the center of mass when discussing the velocity in the linear impulse momentum equations.

$$J_x=mv_{f,x}-mv_{i,x} \tag{14.1.3}$$

$$J_y=mv_{f,y}-mv_{i,y} \tag{14.1.4}$$

$$K_z=I_G\omega_f-I_G\omega_i \tag{14.1.5}$$

Impulse:

As discussed with particles, a linear impulse in its most basic form is a force integrated over a time. For a force with a constant magnitude, we can find the magnitude of the impulse by multiplying the magnitude of the force by the time that force is exerted. If the force is not constant, we simply integrate the force function over the set time period. The direction of the impulse vector will be the direction of the force vector, and the units will be a force multiplied by a time (Newton-seconds or pound-seconds, for example).

$$\text{Constant magnitude force:} \quad \vec{J}=\vec{F}*t \tag{14.1.6}$$

$$\text{Non-constant magnitude force:} \quad \vec{J}=\int\vec{F}(t)\,dt \tag{14.1.7}$$

An **angular impulse** is similar to a linear impulse, except it is the **moment exerted over time** instead of the force exerted over time.

$$\text{Constant magnitude moment:} \quad \vec{K}=\vec{M}*t \tag{14.1.8}$$

$$\text{Non-constant magnitude moment:} \quad \vec{K}=\int\vec{M}(t)\,dt \tag{14.1.9}$$

This moment can come either in the form of a torque directly applied to a body, or an off-center force causing a linear and angular impulse at the same time. All moments should be taken about the center of mass of the body.

Figure 14.1.1: If we have two identical spheres with forces as shown above, the force on the left would cause only a linear impulse over time, while the force on the right would cause both a linear impulse and an angular impulse over time.

Momentum:

As discussed with particles, the linear momentum of a body will be equal to the mass of the body times its current velocity. Since velocity is a vector, the momentum will also be a vector, having both magnitude and a direction. Unlike the impulse, which happens over some set time, the momentum is captured as a snapshot of a specific instant in time (usually right before and after some impulse is exerted). The units for linear momentum will be mass times unit distance per unit time. This is usually kilogram-meters per second in metric, or slug-feet per second in English units.

Angular momentum, on the other hand, will be equal to the body's mass moment of inertia (about its center of mass) times its current angular velocity. It's important to note that while the mass moment of inertia remains constant if the body does not change shape, a shape that does change shape will likely have changes in mass moments of inertia along with changes in angular velocity.

Figure 14.1.2: During a spin, figure skaters will often draw their arms and legs in towards their bodies to reduce their mass moment of inertia. With minimal impulses to change the angular momentum, the angular velocity will increase so that angular momentum is conserved. Public Domain image by deerstop.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/vPxE2AFXADE.

Practice 14.1.1

A miter saw has an operating speed of 1500 rpm. The blade and motor armature have a combined weight of 3 lbs and a radius of gyration of 1 inch.

- What is the time required for the bearing friction alone (torque=0.015 inch-lbs) to stop the blade?
- What is the torque a braking system would need to apply to stop the blade in just 0.25 seconds?

Figure 14.1.3: A man cuts wood with a miter saw. Public domain image by John F. Looney

Solution (not yet available):

Not yet available.

Practice 14.1.2

A bowling ball is modeled as a 7-kilogram uniform sphere, 300 mm in diameter. The ball is released with an initial velocity of 6 m/s on a horizontal wooden floor (μ_k = 0.1) with zero angular velocity.

- How long does it take before the ball begins to roll without slipping?
- What is the linear velocity of the ball at this time?

Figure 14.1.4: A bowling ball is released down a lane. Public domain image by Jerry Saslav.

Solution (not yet available):

Not yet available.

14.2: Rigid Body Surface Collisions

In instances where a moving rigid body impacts a solid and immovable surface, the resulting impact can seem chaotic. However, we can still use the ideas of impulse and momentum to predict this impact behavior.

Just as with particles, an important first step in solving problems where a rigid body impacts a surface is to identify the normal and tangential directions. The normal direction will be perpendicular to the surface being impacted, while the tangential direction will be parallel to the surface being impacted. Also important for rigid body impact is identifying both the center of mass for the body and the point of impact between the body and the surface.

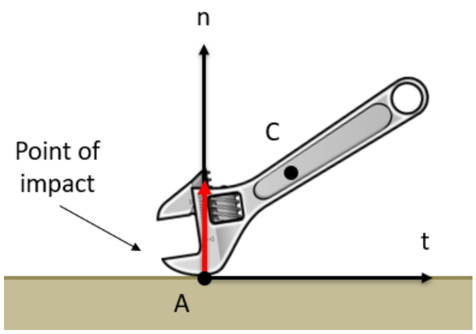


Figure 14.2.1: Dropping a wrench on a concrete floor is a good example of rigid body impact with a solid surface. The first step in solving these problems is to identify the normal and tangential directions, which will be perpendicular and parallel to the surface, respectively. Also important is identifying the center of mass (C) and point of impact (A).

To predict the linear velocities after impact in the n and t directions, as well as the angular velocity after the impact (a total of three unknowns), we will need three equations. These equations will come in the form of the conservation of momentum in the t direction, as well as two equations based on the coefficient of restitution for the velocity in the n direction and the angular velocity after the impact.

For momentum, we will notice that the normal forces during impact will always be in the normal direction. Assuming negligible friction forces (which would be in the tangential direction), we will have no change in momentum in the tangential direction and therefore **no change in velocity in the tangential direction for the center of mass of the body**. This gives us the first equation we can use.

$$v_{t,C,f} = v_{t,C,i} \tag{14.2.1}$$

Next, examining the coefficient of restitution will give us another equation. Specifically, the coefficient of restitution relates the **velocities before an after the collision in the normal direction at the point of impact**.

$$\epsilon = -\frac{v_{n,A,f}}{v_{n,A,f}} \tag{14.2.2}$$

If the collision is elastic we would also conserve the kinetic energy of the body, giving us our third equation. More generally, however, the coefficient of restitution can be used to quantify the amount of energy lost on the collision, with more elastic collisions conserving a higher percentage of kinetic energy and more inelastic equations conserving less kinetic energy. The equation below can be used for any elastic or semi elastic collision with a surface. Using the third equation below, we simply set ϵ equal to one for elastic collisions or equal to the coefficient of restitution for semi-elastic collisions.

$$\epsilon^2 = \frac{KE_f}{KE_i} \tag{14.2.3}$$

Between the three equations above and whatever relevant kinematics relationships are necessary, we should be able to solve for up to three unknowns. This allows us to completely predict the velocities after an impact, assuming we know the pre-impact velocities and the coefficient of restitution for the impact itself.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/LQKk2WLMe4A.

Practice 14.2.1

An 80-centimeter-long metal bar with a mass of 1 kilogram, falling at 2 meters per second, strikes the edge of a table as shown below. Assuming a coefficient of restitution of 0.9, what is the expected velocity and angular velocity of the bar after impact?

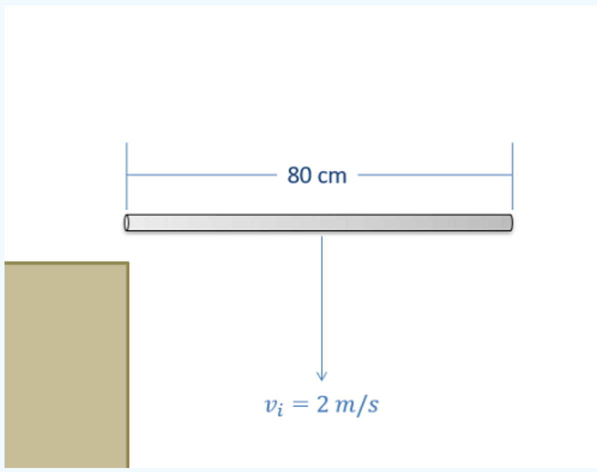


Figure 14.2.2: problem diagram for Example Problem 14.2.2. A horizontal bar falls straight down, until its leftmost end strikes the top of a flat table.

Solution (not yet available)

Not yet available.

14.3: Chapter 14 Homework Problems

Exercise 14.3.1

A flywheel with a diameter of 2 ft and a weight of 60 lbs is rotating at a rate of 600 rpm. A brake applies a friction force to the outer rim of the flywheel, bringing it to a stop in 1.5 seconds. Based on this information, what was the average friction force applied by the brake over this time?

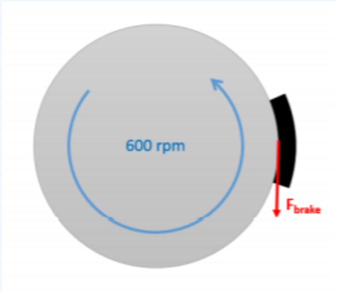


Figure 14.3.1: problem diagram for Example 14.3.1. A brake that is fixed in place applies a tangent frictional force to the outer rim of a rotating flywheel wheel.

Solution

$$F_{brake} = 39.01\ lbs$$

Exercise 14.3.2

A ring-shaped space station can be approximated as a thin ring 60 meters in diameter with a mass of 500,000 kg. Centrifugal acceleration of the spinning station will be used to simulate gravity.

- To simulate the acceleration of Earth (9.81 m/s²), how fast will the station need to be spinning?
- If two thrusters each capable of exerting 10 kN of force will be used to get the station up to this speed, how long will we need to run the thrusters?

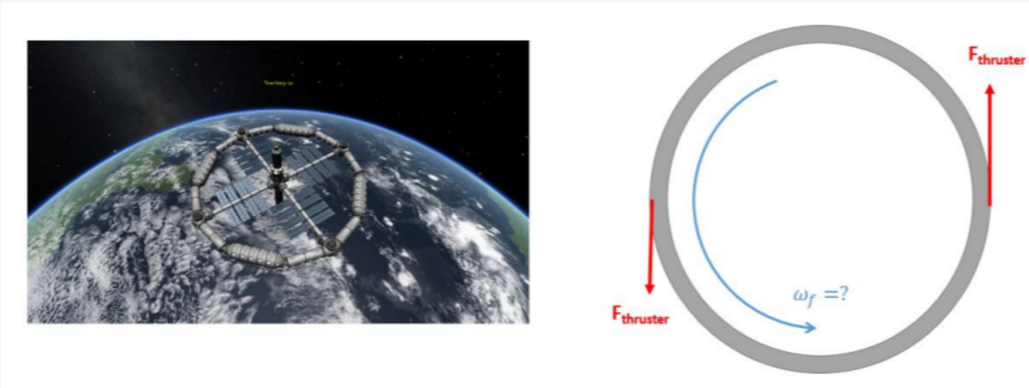


Figure 14.3.2: problem diagram for Exercise 14.3.2. A ring-shaped space station is rotating counterclockwise, due to thrust forces in opposite directions provided by two thrusters on opposite sides of the ring.

Solution

$$\omega_f = 0.571\ \frac{rad}{s}$$

$$t_{thrust} = 428.25\ s$$

CHAPTER OVERVIEW

15: Vibrations with One Degree of Freedom

15.1: Undamped Free Vibrations

Vibrations occur in systems that attempt to return to their resting or equilibrium state when perturbed, or pushed away from their equilibrium state. The simplest vibrations to analyze are undamped, free vibrations with one degree of freedom.

"Undamped" means that there are no energy losses with movement (whether the losses are intentional, from adding dampers, or unintentional, through drag or friction). An undamped system will vibrate forever without any additional applied forces. A simple pendulum has very low damping, and will swing for a long time before stopping. "Damped" means that there are resistive forces and energy losses with movement that cause the system to stop moving eventually.

"Free" means that, after the initial perturbation, the only forces acting on the system are internal to the system (springs, dampers) and/or gravity. A tuning fork continues to vibrate after the one initial perturbation of being struck. In contrast, "forced" means there is an external, typically periodic, force acting on the system. A jackhammer vibrates due to having a supply of compressed air continually forcing the bit up and down, and it stops vibrating very quickly without that external periodic forcing.

"One degree of freedom" means that we will only consider systems with one mass vibrating along one direction (e.g. use variable x) or about one axis (e.g. use variable θ). Systems having more than one mass or vibrating along or about two or more axes have more than one degree of freedom.

We can derive the equation of the system by setting up a free body diagram. Consider a mass sitting on a frictionless surface, attached to a wall via a spring.

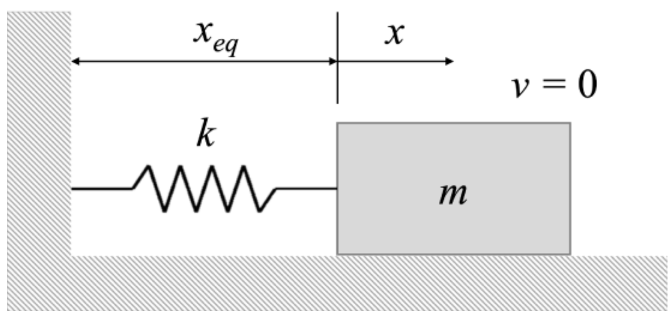


Figure 15.1.1: This is a system consisting of a mass attached to the wall via a spring, sitting on a frictionless surface. The system is currently in equilibrium, and the spring is not stretched or compressed.

The system above is in equilibrium. It is at rest, and will stay at rest unless some other force acts on it.

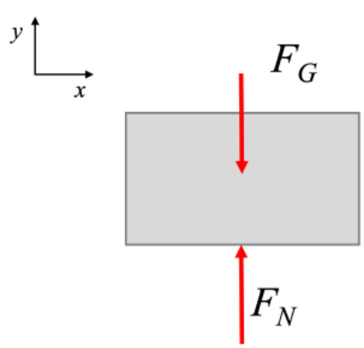


Figure 15.1.2: Free body diagram of the system in equilibrium position. Since the spring is at its unstretched length, it does not produce a force.

To start the system vibrating, we need to perturb it. Perturbation is moving the system away from equilibrium by a small amount.

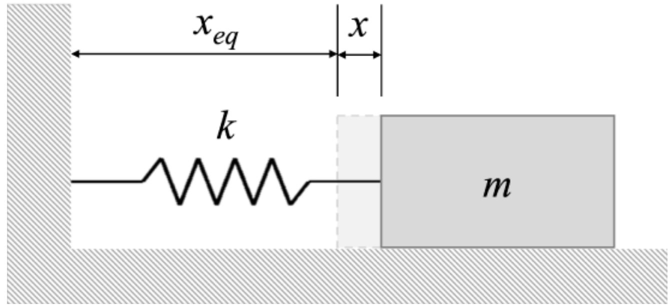


Figure 15.1.3: Mass-spring system perturbed by an amount $+x$ from equilibrium. This stretches the spring, causing a spring force which tends to pull the mass back toward equilibrium.

When we perturb this system, we either stretch or compress the spring. This generates a spring force, and the spring force is always in a direction that tends to pull the system back toward equilibrium.

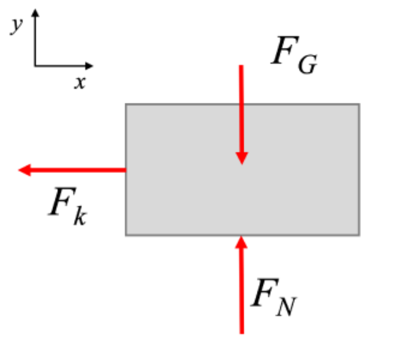


Figure 15.1.4: Free body diagram of the mass-spring system perturbed by an amount $+x$ from equilibrium. This stretches the spring, causing a spring force which tends to pull the mass back toward equilibrium.

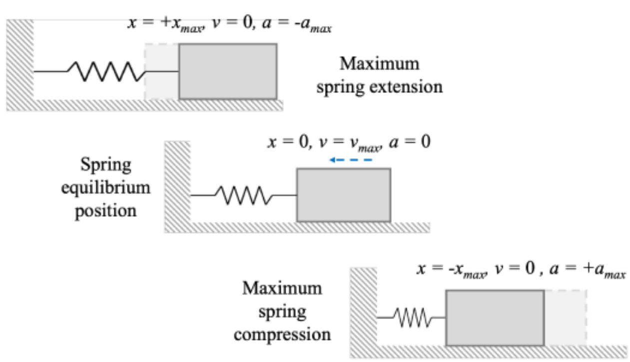


Figure 15.1.5: Mass-spring system vibration. After an initial perturbation, the system oscillates between two extreme positions ($-x_{max}$ and x_{max}). The extreme positions are turnaround points where the velocity is zero, but the spring force is at a maximum (maximum stretch/compression) and therefore the acceleration is maximum. As the system passes through the equilibrium position, the spring is no longer stretched (zero force, zero acceleration), but the velocity is at its maximum (this can be determined using conservation of energy) and the inertia of the system carries the mass past the equilibrium position. In the absence of damping or the application of another force, the system will oscillate forever.

We can generate the equation of motion of the system, and determine the specifics of how it will vibrate, by analyzing this perturbed state. Recall that the spring force or moment is:

$$\vec{F}_k = k\vec{x} \tag{15.1.1}$$

$$\vec{M}_k = k\vec{\omega} \tag{15.1.2}$$

Note that the spring constants in the above equations have different units, depending on whether the spring is linear (Newton/meter) or torsional (Newton-meter/radian), and that θ must be given in radians. The magnitude of the spring force depends on x, the distance perturbed **from the spring's unstretched length** (not necessarily the equilibrium position of the system), and the same is true for the moment of a torsional spring. The spring force or moment is in the direction/orientation opposite that of the displacement. That is, if you pull the mass to the right, the spring force points to the left.

The process for finding the equation of motion of the system is as follows:

1. Sketch the system with a small positive perturbation (x or θ).
2. Draw the free body diagram of the perturbed system. Ensure that the spring force has a direction opposing the perturbation.
3. Find the one equation of motion for the system in the perturbed coordinate using Newton's Second Law. Keep the same positive direction for position, and assign positive acceleration in the same direction.
4. Move all terms of the equation to one side, and check that all terms are positive. If all terms are not positive, there is an error in the direction of displacement, acceleration, and/or spring force.

For the example system above, with mass m and spring constant k, we derive the following:

$$\sum F_x = ma_x = m\ddot{x} \tag{15.1.3}$$

$$-F_k = m\ddot{x} \tag{15.1.4}$$

$$-kx = m\ddot{x} \tag{15.1.5}$$

$$m\ddot{x} + kx = 0 \tag{15.1.6}$$

This gives us a differential equation that describes the motion of the system. We can rewrite it in normal form:

$$m\ddot{x} + kx = 0 \tag{15.1.7}$$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0 \tag{15.1.8}$$

$$\Rightarrow \ddot{x} + \omega_n^2 x = 0 \tag{15.1.9}$$

The term ω_n is called the **angular natural frequency of the system**, and has units of radians/second.

$$\omega_n^2 = \frac{k}{m} \tag{15.1.10}$$

$$\omega_n = \sqrt{\frac{k}{m}} \tag{15.1.11}$$

Assuming that the initial perturbation of the system can be described by the position and velocity of the mass at $t = 0$, then:

$$x(0) = x_0 \tag{15.1.12}$$

$$v(0) = \dot{x}(0) = v_0. \tag{15.1.13}$$

The solution to the differential equation, that provides the position $x(t)$ of the system at time t, is:

$$x(t) = C\sin(\omega_n t + \phi), \tag{15.1.14}$$

$$\text{where:} \quad \omega_n = \sqrt{\frac{k}{m}},\ C = \sqrt{\left(\frac{v_0}{\omega_n}\right)^2 + x_0^2},\ \phi = \tan^{-1}\left(\frac{x_0\omega_n}{v_0}\right). \tag{15.1.15}$$

The **amplitude** C describes the maximum displacement during the oscillations (i.e. x_{max}), and the **phase** ϕ describes how the sine function is shifted in time.

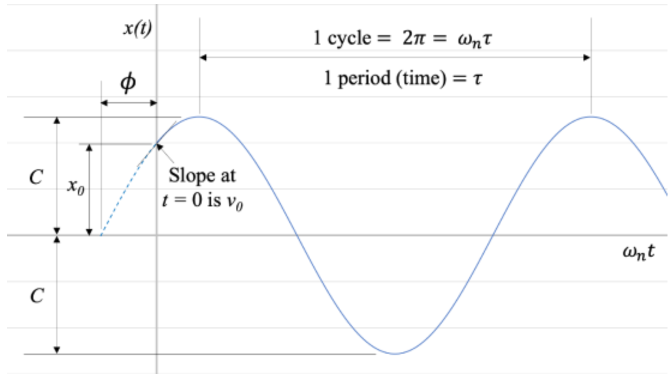


Figure 15.1.6: Displacement response of the mass spring system (solution to the differential equation).

For a system where there is torsional vibration (that is, the oscillation involves a rotation), the equations are similarly:

$$I\ddot{\theta}+k\theta=0 \tag{15.1.16}$$

$$\Rightarrow \ddot{\theta}+\frac{k}{I}\theta=0 \tag{15.1.17}$$

$$\Rightarrow \ddot{\theta}+\omega_n^2\theta=0, \tag{15.1.18}$$

$$\text{where } \omega_n=\sqrt{\frac{k}{I}}. \tag{15.1.19}$$

Assuming that the initial perturbation of the system can be described by the position and velocity of the mass at $t=0$,

$$x(0)=x_0 \tag{15.1.20}$$

$$v(0)=\dot{x}(0)=v_0. \tag{15.1.21}$$

The solution to the differential equation, that provides the position $x(t)$ of the system at time t, is:

$$x(t)=C\sin(\omega_n t+\phi) \tag{15.1.22}$$

$$\text{where:}\quad \omega_n=\sqrt{\frac{k}{I}},\; C=\sqrt{\left(\frac{\omega_0}{\omega_n}\right)^2+\theta_0^2},\; \phi=\tan^{-1}\left(\frac{\theta_0\omega_n}{\omega_0}\right). \tag{15.1.23}$$

Example 15.1.1

Find an expression for the angular natural frequency of the following system, and find the maximum amplitude of vibration of the system with mass m = 10 kg and spring constant k = 200 N/m when given an initial displacement of x_0 = 0.1 m and an initial velocity of v_0 = 0.3 m/s.

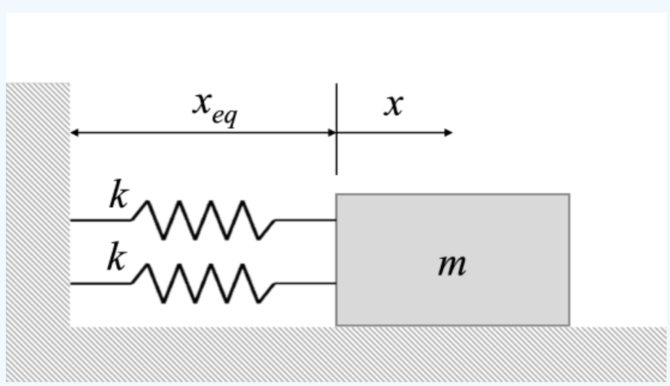


Figure 15.1.7: problem diagram for Example 15.1.1. A rectangular mass on a flat surface has its left edge attached to two identical springs, whose other ends are attached to a wall.

Solution

Video 15.1.1: Worked solution to example problem 15.1.1. YouTube source: https://youtu.be/J1TVxxVjV_c.

Example 15.1.2

Determine the equation of motion of the system from Newton's Second Law. Assume mass m = 5 kg and spring constant k = 500 N/m. Find the initial displacement, x_0, such that the mass oscillates over a total range of 4 meters. Assume the initial perturbation velocity, v_0, is 10 m/s.

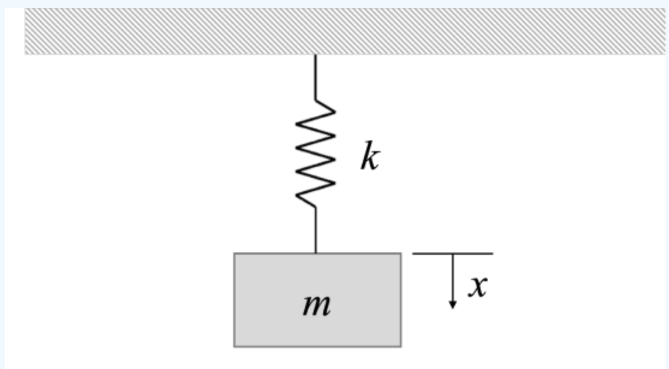


Figure 15.1.8: problem diagram for Example 15.1.2. A mass hangs from a spring attached to a ceiling.

Solution

Video 15.1.2: Worked solution to example problem 15.1.2. YouTube source: https://youtu.be/TAy412iVvwE.

15.2: Viscous Damped Free Vibrations

Viscous damping is damping that is proportional to the velocity of the system. That is, the faster the mass is moving, the more damping force is resisting that motion. Fluids like air or water generate viscous drag forces.

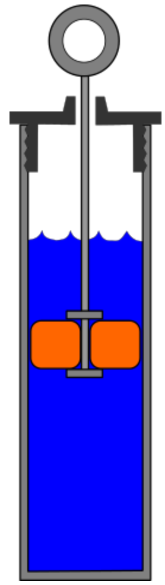


Figure 15.2.1: A diagram showing the basic mechanism in a viscous damper. As the system (mass) attached to the loop at the top vibrates up and down, the damper will resist motion in both directions due to the piston passing through the fluid. Image by Egmason, CC-BY SA.

We will only consider linear viscous dampers, that is where the damping force is linearly proportional to velocity. The equation for the force or moment produced by the damper, in either x or θ, is:

$$\vec{F}_c = c\dot{\vec{x}}, \tag{15.2.1}$$

$$\vec{M}_c = c\dot{\vec{\theta}}, \tag{15.2.2}$$

where c is the damping constant. This is a physical property of the damper based on the type of fluid, size of the piston, etc. Note that the units of c change depending on whether it is damping linear motion (N-s/m) or rotational motion (N-m s/rad).

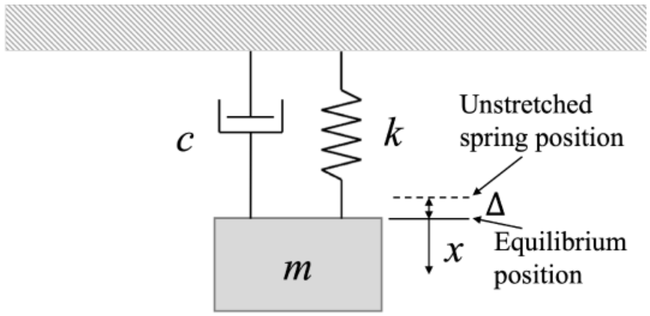


Figure 15.2.2: Diagram of a hanging mass-spring system, with a linear viscous damper, in equilibrium position. The spring is stretched from its natural length.

When the system is at rest in the equilibrium position, the damper produced no force on the system (no velocity), while the spring can produce force on the system, such as in the hanging mass shown above. Recall that this is the equilibrium position, but the spring is NOT at its unstretched length, as the static mass produces an extention of the spring.

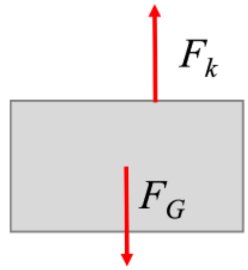


Figure 15.2.3: Free body diagram of the system in equilibrium position. The spring is at its equilibrium position, but it is stretched and does produce a force.

If we perturb the system (applying an initial displacement, an initial velocity, or both), the system will tend to move back to its equilibrium position. What that movement looks like will depend on the system parameters (m, c, and k).

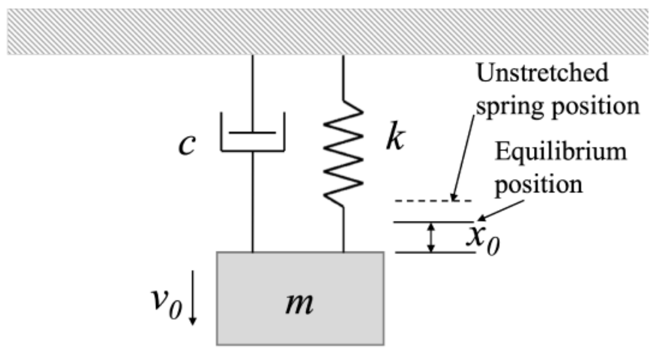


Figure 15.2.4: The system in a perturbed position. The spring is stretched further and the damper is extended, compared to their equilibrium positions.

To determine the equation of motion of the system, we draw a free body diagram of the system with perturbation and apply Newton's Second Law.

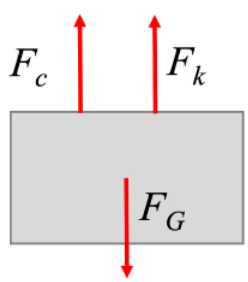


Figure 15.2.5: Free body diagram of the system with perturbation.

The process for finding the equation of motion of the system is again:

1. Sketch the system with a small positive perturbation (x or θ).
2. Draw the free body diagram of the perturbed system. Ensure that the spring force and the damper force have directions opposing the perturbation.
3. Find the one equation of motion for the system in the perturbed coordinate using Newton's Second Law. Keep the same positive direction for position, and assign positive acceleration in the same direction.
4. Move all terms of the equation to one side, and check that all terms are positive. If all terms are not positive, there is an error in the direction of displacement, acceleration, and/or spring or damper force.

For the example system above, with mass m, spring constant k and damping constant c, we derive the following:

$$\sum F_x = ma_x = m\ddot{x} \tag{15.2.3}$$

$$-F_k - F_c = m\ddot{x} \tag{15.2.4}$$

$$-kx - c(\dot{x}) = m\ddot{x} \tag{15.2.5}$$

$$m\ddot{x} + c(\dot{x}) + kx = 0 \tag{15.2.6}$$

This gives us a differential equation that describes the motion of the system. We can rewrite it in normal form:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{15.2.7}$$

$$\Rightarrow \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{15.2.8}$$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{15.2.9}$$

As before, the term ω_n is called the **angular natural frequency of the system**, and has units of rad/s.

$$\omega_n^2 = \frac{k}{m}; \quad \omega_n = \sqrt{\frac{k}{m}} \tag{15.2.10}$$

ζ (zeta) is called the damping ratio. It is a dimensionless term that indicates the level of damping, and therefore the type of motion of the damped system.

$$\zeta = \frac{c}{c_c} = \frac{\text{actual damping}}{\text{critical damping}} \tag{15.2.11}$$

The expression for critical damping comes from the solution of the differential equation. The solution to the system differential equation is of the form

$$x(t) = ae^{rt}, \tag{15.2.12}$$

where a is constant and the value(s) of r can be can be obtained by differentiating this general form of the solution and substituting into the equation of motion.

$$mr^2e^{rt} + cre^{rt} + ke^{rt} = 0 \tag{15.2.13}$$

$$\Rightarrow (mr^2 + cr + k)e^{rt} = 0 \tag{15.2.14}$$

Because the exponential term is never zero, we can divide both sides by that term and get:

$$mr^2 + cr + k = 0. \tag{15.2.15}$$

Using the quadratic formula, we can find the roots of the equation:

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \tag{15.2.16}$$

Critical damping occurs when the term under the square root sign equals zero:

$$c_c^2 = 4mk \tag{15.2.17}$$

$$c_c = 2\sqrt{mk} = 2m\omega_n \tag{15.2.18}$$

Four Viscous Damping Cases:

There are four basic cases for the damping ratio. For the solutions that follow in each case, we will assume that the initial perturbation displacement of the system is x_0 and the initial perturbation velocity of the system is v_0.

1. $\zeta = 0$: Undamped

$$c = 0 \tag{15.2.19}$$

This is the case covered in the previous section. Undamped systems oscillate about the equilibrium position continuously, unless some other force is applied.

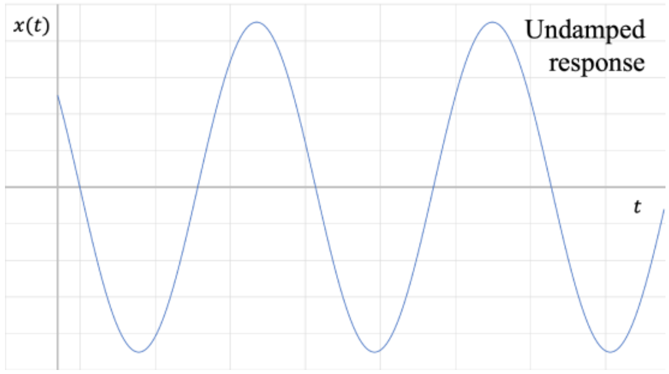


Figure 15.2.6: Response of an undamped system.

2. $\zeta > 1$: Overdamped

$$c^2 > 4mk \tag{15.2.20}$$

Roots are both real and negative, but not equal to each other. Overdamped systems move slowly toward equilibrium without oscillating.

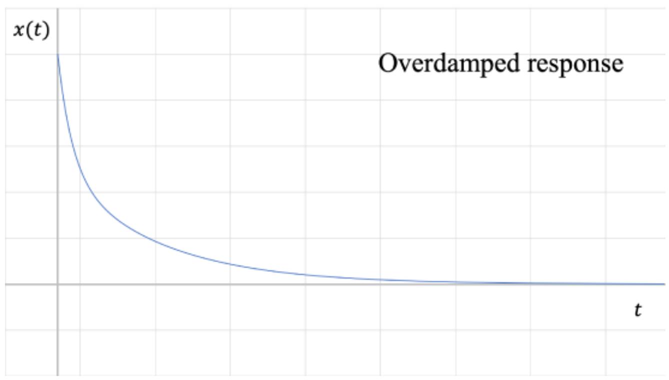


Figure 15.2.7: Response of an overdamped system.

The response for an overdamped system is:

$$x(t) = a_1 e^{\left(\frac{-c+\sqrt{c^2-4mk}}{2m}\right)t} + a_2 e^{\left(\frac{-c-\sqrt{c^2-4mk}}{2m}\right)t}, \tag{15.2.21}$$

$$\text{where } a_1 = \frac{-v_0 + r_2 x_0}{r_2 - r_1} \text{ and } a_2 = \frac{v_0 + r_1 x_0}{r_2 - r_1}. \tag{15.2.22}$$

3. $\zeta = 1$: Critically damped

$$c^2 = 4mk (= c_c^2) \tag{15.2.23}$$

Roots are real and both equal to $-\omega_n$. Critically-damped systems will allow the fastest return to equilibrium without oscillation.

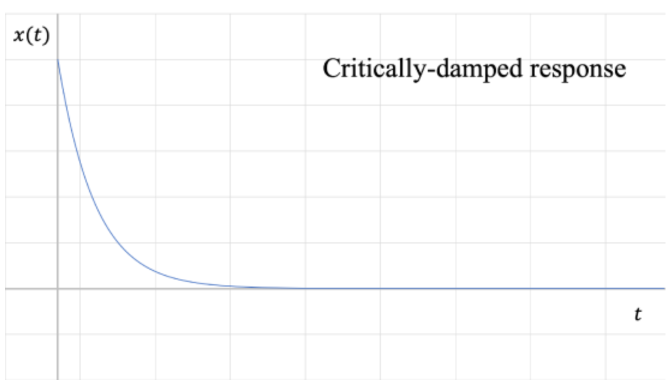


Figure 15.2.8: Response of an critically-damped system.

The solution for a critically-damped system is:

$$x(t) = (A + Bt)e^{-\omega_n t}, \tag{15.2.24}$$

$$\text{where } A = x_0 \text{ and } B = v_0 + x_0\omega_n. \tag{15.2.25}$$

4. $\zeta < 1$: Underdamped

$$c^2 < 4mk \tag{15.2.26}$$

The roots are complex numbers. Underdamped systems do oscillate around the equilibrium point; unlike undamped systems, the amplitude of the oscillations diminishes until the system eventually stops moving at the equilibrium position.

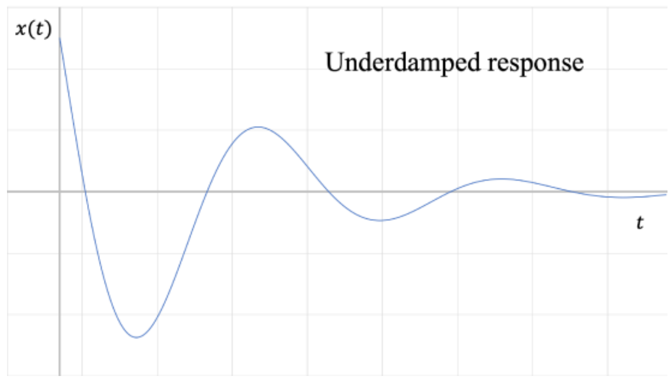


Figure 15.2.9: Response of an underdamped system.

The solution for an underdamped system is:

$$x(t) = [C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)]e^{-\omega_n \zeta t}, \tag{15.2.27}$$

$$\text{where } C_1 = \frac{v_0 + \omega_n \zeta x_0}{\omega_d}, \ C_2 = x_0, \text{ and } \zeta = \frac{c}{2m\omega_n}. \tag{15.2.28}$$

ω_d is called the **damped natural frequency** of the system. It is always less than ω_n:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}. \tag{15.2.29}$$

The period of the underdamped response differs from the undamped response as well.

$$\text{Undamped:} \quad \tau_n = \frac{2\pi}{\omega_n} \tag{15.2.30}$$

$$\text{Underdamped:} \quad \tau_d = \frac{2\pi}{\omega_d} \tag{15.2.31}$$

Comparison of Viscous Damping Cases:

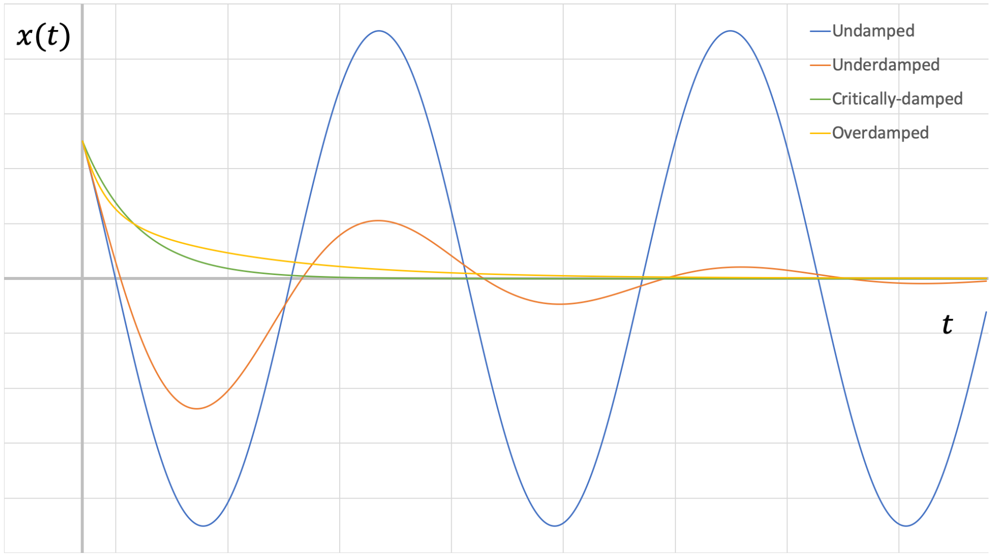


Figure 15.2.10: Responses for all four types of system (or values of damping ratio) in viscous damping. All four systems have the same mass and spring values, and have been given the same initial perturbations (initial position and initial velocity); this is apparent because they start at the same y-intercept and have the same slope at $x = 0$.

In the figure above, we can see that the critically-damped response results in the system returning to equilibrium the fastest. Also, we can see that the underdamped system amplitude is quite attenuated compared to the undamped case.

15.3: Friction (Coulomb) Damped Free Vibrations

Friction can also provide vibration damping. In this case, however, the damping is **not** proportional to the magnitude of velocity. It only depends on the direction of velocity.

We remember from the section on dry friction that the force of friction in sliding depends only on the coefficient of kinetic friction, μ_k, and the normal force, F_N.

$$F_f = \mu_k F_N \tag{15.3.1}$$

The above equation does not include velocity. We know that kinetic friction acts to oppose motion, however, so a more complete expression would be:

$$F_f = -\operatorname{sgn}(\dot{x})\mu_k F_N, \tag{15.3.2}$$

where sgn is the "sign" function, a function that captures the sign (direction) of velocity. The above equation then indicates that the direction of friction is always opposite the direction of velocity, but the magnitude of velocity does not make a difference in the magnitude of friction.

The equation of motion of the system becomes:

$$m\ddot{x} + \mu mg\,\operatorname{sgn}(\dot{x}) + kx = 0, \tag{15.3.3}$$

and the solution to this equation of motion is:

$$x(t) = \left(x_0 - \frac{(2n-1)\mu mg}{k}\right)\cos(\omega_n t) + \frac{\mu mg}{k}(-1)^{n+1}. \tag{15.3.4}$$

If we plot the response, we can see that there are several differences from a system with viscous damping.

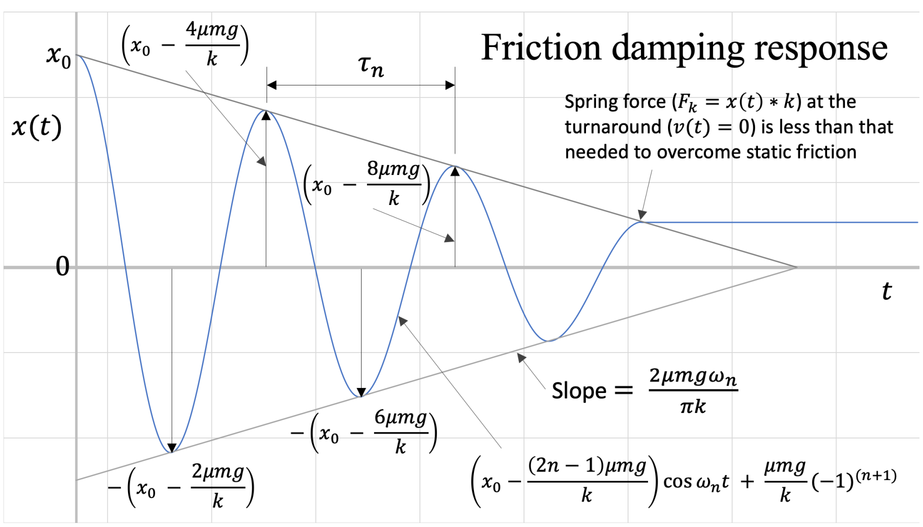


Figure 15.3.1: Response of the system in friction damping.

Some differences when compared to viscous damping include:

1. The system oscillates at the natural frequency of the system, not a damped natural frequency.
2. The bounding curves are linear, not exponential.
3. The system does not return to zero. This is because the magnitude of the friction force does not diminish as the system amplitude reduces, and at some point the spring force is no longer able to overcome the static friction that the system experiences when it changes direction ($v = 0$).

Comparison to Viscous Underdamped System

If we consider our simple linear mass-spring system, the magnitude of F_f does not change with velocity, unlike with viscous damping. If we plotted both types of damping for the same system, we would get the following:

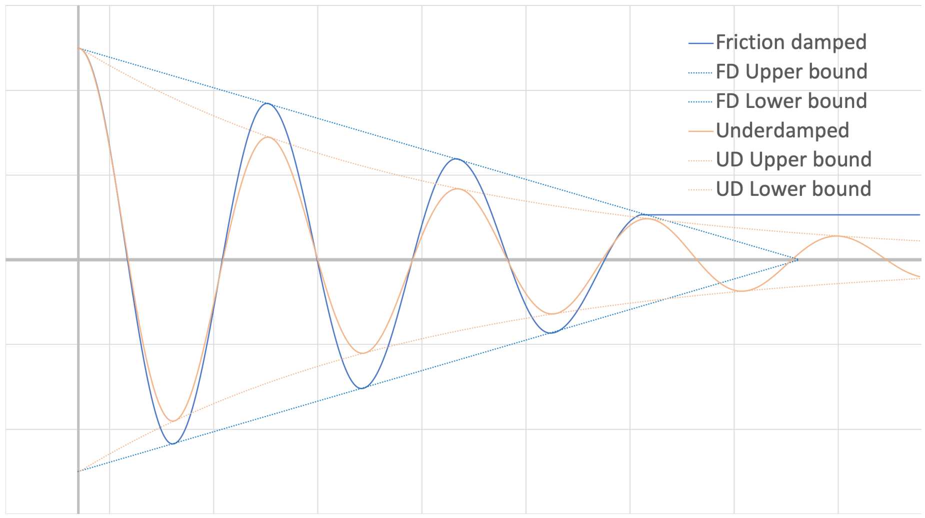


Figure 15.3.2: Response of the system in friction damping and in viscous damping, for the same initial conditions (x_0, v_0), spring constants and masses.

Note that the viscous damping has more reduction in amplitude earlier (despite relatively light damping), but continues oscillating past the point when the friction-damped system has stopped (specific relative values are dependent on the values of damping constant and coefficients of friction). Also note that since the viscous damping is relatively light, the difference in period between the two plots is quite small in this example.

15.4: Undamped Harmonic Forced Vibrations

Often, mechanical systems are not undergoing free vibration, but are subject to some applied force that causes the system to vibrate. In this section, we will consider only harmonic (that is, sine and cosine) forces, but any changing force can produce vibration.

When we consider the free body diagram of the system, we now have an additional force to add: namely, the external harmonic excitation.

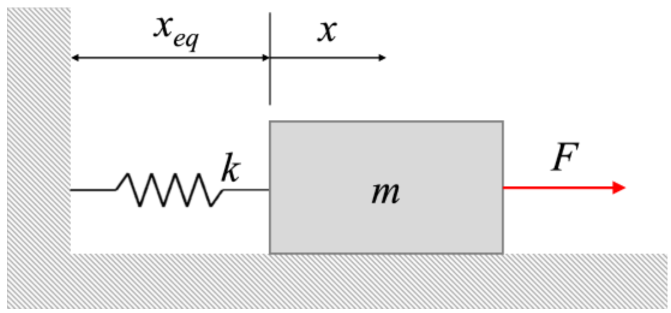


Figure 15.4.1: A mass-spring system with an external force, F, applying a harmonic excitation.

The equation of motion of the system above will be:

$$m\ddot{x} + kx = F, \tag{15.4.1}$$

where F is a force of the form:

$$F = F_0 \sin(\omega_0 t). \tag{15.4.2}$$

This equation of motion for the system can be re-written in standard form:

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin(\omega_0 t). \tag{15.4.3}$$

The solution to this system consists of the superposition of two solutions: a particular solution, x_p (related to the forcing function), and a complementary solution, x_c (which is the solution to the system without forcing).

As we saw previously, the complementary solution is the solution to the undamped free system:

$$x_c = C\sin(\omega_n t + \phi). \tag{15.4.4}$$

We can obtain the particular solution by assuming a solution of the form:

$$x_p = D\sin(\omega_0 t), \tag{15.4.5}$$

where ω_0 is the frequency of the harmonic forcing function. We differentiate this form of the solution, and then sub into the above equation of motion:

$$\ddot{x}_p = -\omega_0^2 D\sin(\omega_0 t) \tag{15.4.6}$$

$$-m\omega_0^2 D\sin(\omega_0 t) + kD\sin(\omega_0 t) = F_0\sin(\omega_0 t) \tag{15.4.7}$$

After solving for D, we can then use it to find the particular solution, x_p:

$$D = \frac{\frac{F_0}{k}}{1 - \left(\frac{\omega_0}{\omega_n}\right)^2} \tag{15.4.8}$$

$$x_p = \frac{\frac{F_0}{k}}{1 - \left(\frac{\omega_0}{\omega_n}\right)^2} \sin(\omega_0 t). \tag{15.4.9}$$

Thus, the general solution for a forced, undamped system is:

$$x_G(t) = \frac{\frac{F_0}{k}}{1 - \left(\frac{\omega_0}{\omega_n}\right)^2} \sin(\omega_0 t) + C \sin(\omega_n t + \phi) \tag{15.4.10}$$

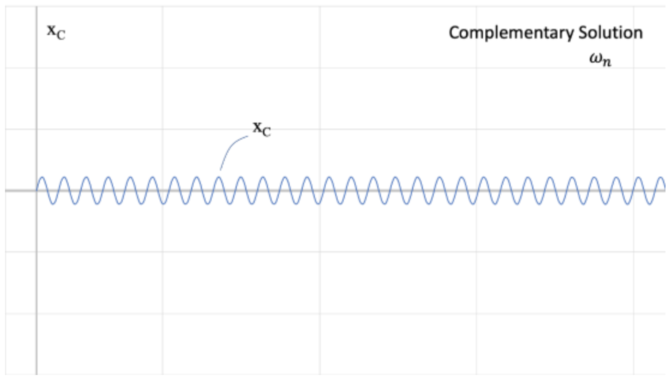


Figure 15.4.2: The complementary solution of the equation of motion. This represents the natural response of the system, and oscillates at the angular natural frequency. This is the transient response.

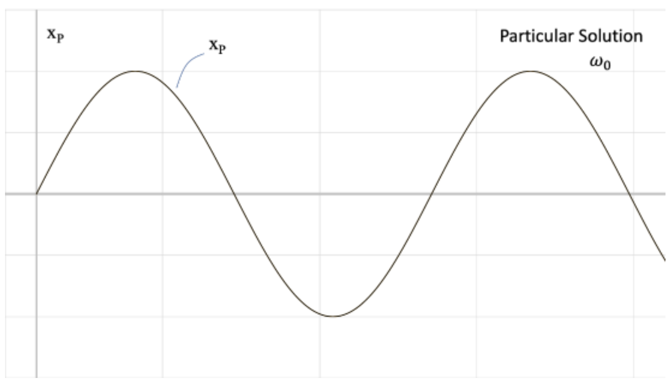


Figure 15.4.3: The particular solution of the equation of motion. This represents the forced response of the system, and oscillates at the angular forced frequency. This is the steady-state response.

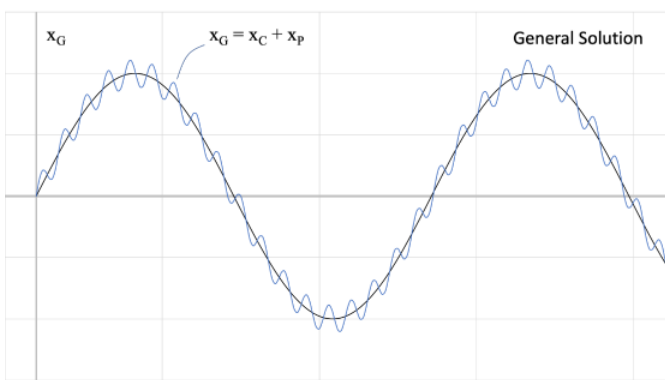


Figure 15.4.4: The general solution of the equation of motion. This represents the combined response of the system, and the sum of the complementary (or natural) and particular (or forced) responses.

The above figures show the two responses at different frequencies. Recall that the value of ω_n comes from the physical characteristics of the system (m, k) and ω_0 comes from the force being applied to the system. These responses are summed, to achieve the blue response (general solution) in Figure $15.4.4$ above.

Steady-State Response:

In reality, this superimposed response does not last long. Every real system has some damping, and the natural response of the system will be damped out. As long as the external harmonic force is applied, however, the response to it will remain. When evaluating the response of the system to a harmonic forcing function, we will typically consider the steady-state response, when the natural response has been damped out and the response to the forcing function remains.

Amplitude of Forced Vibration

The amplitude of the steady-state forced vibration depends on the ratio of the forced frequency to the natural frequency. As ω_0 approaches ω_n (the ratio approaches 1), the magnitude D becomes very large. We can define a magnification factor:

$$MF = \frac{\dfrac{\dfrac{F_0}{k}}{1-\left(\dfrac{\omega_0}{\omega_n}\right)^2}}{\dfrac{F_0}{k}} = \frac{1}{1-\left(\dfrac{\omega_0}{\omega_n}\right)^2} \tag{15.4.11}$$

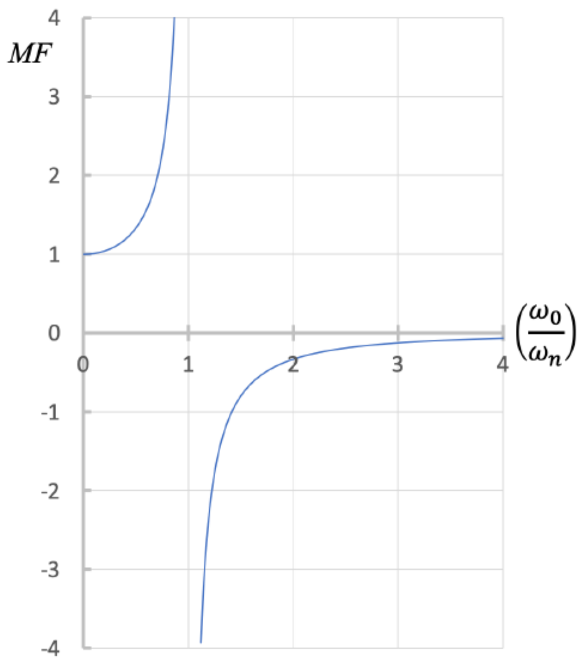


Figure 15.4.5: The magnification factor, MF, is defined as the ratio of the amplitude of the steady-state vibration to the displacement that would be achieved by static deflection.

From the figure above, we can discuss various cases:

- $\omega_0 = \omega_n$: Resonance occurs. This results in very large-amplitude vibrations, and is associated with high stress and failure to the system.
- $\omega_0 \sim 0, \mathbf{MF} \sim 1$: The forcing function is nearly static, leaving essentially the static deflection and limited natural vibration.
- $\omega_0 < \omega_n$: Magnification is positive and greater than 1, meaning the vibrations are in phase (when the force acts to the left, the system displaces to the left) and the amplitude of vibration is larger than the static deflection.
- $\omega_0 > \omega_n$: Magnification is negative and the absolute value is typically smaller than 1, meaning the vibration is out of phase with the motion of the forcing function (when the force acts to the left, the system displaces to the right) and the amplitude of vibration is smaller than the static deflection.
- $\omega_0 >> \omega_n$: The force is changing direction too fast for the block's motion to respond.

Rotating Unbalance:

One common cause of harmonic forced vibration in mechanical systems is rotating unbalance. This occurs when the axis of rotation does not pass through the center of mass, meaning that the center of mass experiences some acceleration instead of remaining stationary. This causes a force on the axle that changes direction as the center of mass rotates. We can represent this as a small mass, m, rotating about the axis of rotation at some distance, called an eccentricity, e. The forced angular frequency, ω_0, in this case is the angular frequency of the rotating system.

? Example 15.4.1

A 10-kg fan is fixed to a lightweight beam. The static weight of the fan deflects the beam by 20 mm. If the blade is designed to spin at ω = 15 rad/s, and the blade is mounted off-center (equivalent to a 1.5 kg mass at 50 mm from the axis of rotation), determine the steady-state amplitude of vibration.

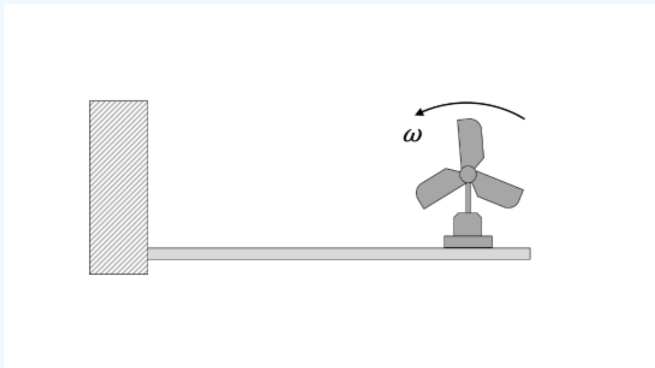


Figure 15.4.6: problem diagram for Example 15.4.1. A horizontal beam not touching the ground is attached to a wall at its left end and holding up a three-bladed rotary fan on its right end.

Solution:

Video 15.4.1: Worked solution to example problem 15.4.1. YouTube source: https://youtu.be/k0vJLxaAjtw.

? Example 15.4.2

You are designing a stylish fan that uses only one blade. Approximate that blade as a narrow plate with a density per unit length of 20 g/cm. The base weight of the rest of the device (except for the blade) is 4 kg, and the whole thing is mounted on a lightweight beam. If the spring constant of the beam is k = 1000 N/m, find the length of blade than will cause resonance if the fan is designed to spin at ω = 15 rad/s.

Figure 15.4.7: problem diagram for Example 15.4.2. A horizontal beam not touching the ground is attached to a wall at its left end and holding up a single-bladed rotary fan on its right.

Solution:

Video 15.4.2: Worked solution to example problem 15.4.2. YouTube source: https://youtu.be/tV4ETwwOyX0.

15.5: Viscous Damped Harmonic Forced Vibrations

As described in the previous section, many vibrations are caused by an external harmonic forcing function (such as rotating unbalance). While we assumed that the natural vibrations of the system eventually damped out somehow, leaving the forced vibrations at steady-state, by explicitly including viscous damping in our model we can evaluate the system through the transient stage when the natural vibrations are present.

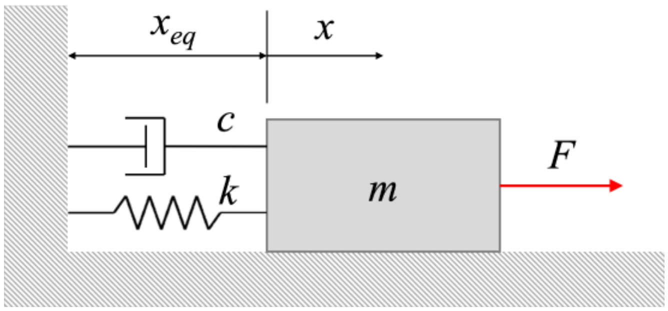


Figure 15.5.1: A mass-spring-damper system with an external force, F, applying a harmonic excitation.

Consider the system above. The equation of the system becomes:

$$m\ddot{x}+c\dot{x}+kx=F_0\sin(\omega_0 t) \tag{15.5.1}$$

$$\Rightarrow \ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=\frac{F_0}{m}\sin(\omega_0 t). \tag{15.5.2}$$

Because the natural vibrations will damp out with friction (as mentioned in undamped harmonic vibrations), we will only consider the particular solution. This particular solution will have the form:

$$x_p(t)=X'\sin(\omega_0 t-\phi'). \tag{15.5.3}$$

After solving, we determine that the expressions for X' and ϕ' are:

$$X'=\frac{\frac{F_0}{k}}{\sqrt{\left[1-\left(\frac{\omega_0}{\omega_n}\right)^2\right]^2+\left[2\frac{c}{c_c}\frac{\omega_0}{\omega_n}\right]^2}} \tag{15.5.4}$$

$$\phi'=\tan^{-1}\left[\frac{2\frac{c}{c_c}\frac{\omega_0}{\omega_n}}{1-\left(\frac{\omega_0}{\omega_n}\right)^2}\right]. \tag{15.5.5}$$

The magnification factor now becomes:

$$MF=\frac{X'}{\frac{F_0}{k}}=\frac{1}{\sqrt{\left[1-\left(\frac{\omega_0}{\omega_n}\right)^2\right]^2+\left[2\frac{c}{c_c}\frac{\omega_0}{\omega_n}\right]^2}} \tag{15.5.6}$$

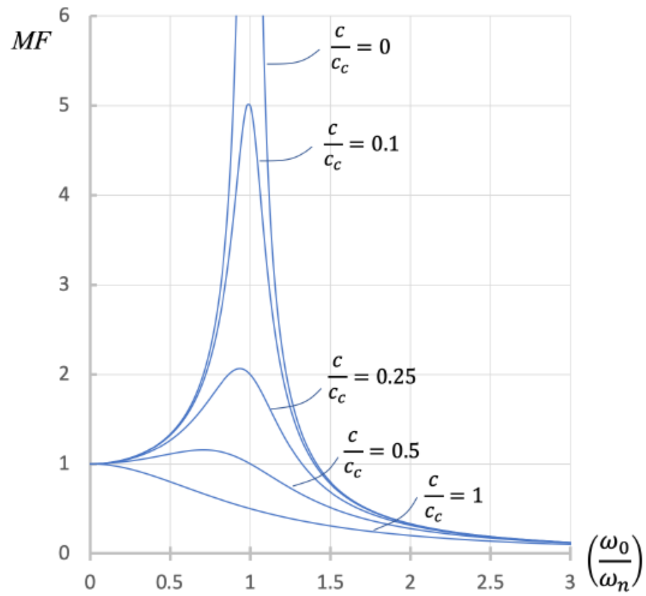


Figure 15.5.2: This figure shows the various magnification factors associated with different levels of (under)damping.

From the figure above, we can see that the extreme amplitudes at resonance only occur when the damping ratio = 0 and the ratio of frequencies is 1. Otherwise, damping inhibits the out-of-control vibrations that would otherwise be seen at resonance.

CHAPTER OVERVIEW

16: Appendix 1 - Vector and Matrix Math

16.1: Vectors

Vectors are used in engineering mechanics to represent quantities that have both a **magnitude** and a **direction**. Many engineering quantities, such as forces, displacements, velocities, and accelerations, will need to be represented as vectors for analysis. Vector quantities contrast with **scalar** values (such as mass, area, or speed), which have a magnitude but no direction.

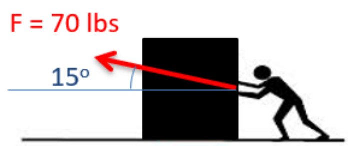


Figure 16.1.1: The force below is represented as a vector. It has both a magnitude and a direction.

When dealing with vectors in equations, engineers commonly denote something as a vector by putting an arrow over the variable. Variables without an arrow over top of them represent a scalar quantity, or simply the magnitude of that vector quantity.

$$\text{Vector Quantity:} \quad \vec{F} \tag{16.1.1}$$

$$\text{Scalar Quantity:} \quad F \tag{16.1.2}$$

In contrast with scalar quantities, we cannot add, subtract, multiply or divide them by simply adding, subtracting, multiplying or dividing the magnitudes. The directions will also play a critical role in solving equations that contain vector quantities.

Vector Representation:

To represent a vector quantity, we will generally have two options. These two options are:

- **Magnitude and Direction Form**: Where the magnitude is given as a single quantity and the direction is given via an angle or combination of angles.
- **Component Form**: Where the magnitude and direction are given through component magnitudes in each coordinate direction.

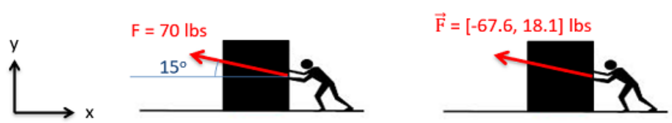


Figure 16.1.2: The drawing above shows a force as vector. On the left the vector is represented as a magnitude and a direction. On the right the same force is represented in terms of its x- and y-components.

The magnitude and direction form of vector quantities are often used at the start and end of a problem. This is because it is often easier to measure things likes forces and velocities as a magnitude and direction at the start of a problem, and it is often easier to visualize the final result as a magnitude and direction at the end. Vectors represented as a magnitude and direction need to be shown visually through the use of an arrow, where the magnitude is the length of the arrow, and the direction is shown through the arrow head and an angle or angles relative to some known axes or other direction.

The component form of a vector is often used in middle of the problem because it is far easier to do math with vector quantities in component form. Vectors can be represented in component form in one of two ways. First we can use square brackets to indicate a vector, with the x, y, and possibly z components separated by commas. Alternatively, we can write out a vector in component form using the magnitudes in front of unit vectors to indicate directions (generally the $\hat{i}$, $\hat{j}$, and $\hat{k}$ unit vectors for the x, y, and z directions respectively). Neither of these component forms relies on a visual depiction of the vector as with the magnitude and direction form, though it is important to clearly identify the coordinate system in earlier diagrams.

$$\text{With Brackets:} \quad \vec{F} = [3, 4, 5] \tag{16.1.3}$$

$$\text{With Unit Vectors:} \quad \vec{F} = 3\hat{i} + 4\hat{j} + 5\hat{k} \tag{16.1.4}$$

Converting Between Vector Representations in 2D

For our analysis, we will often find it advantageous to have the vectors in one form or the other, and will therefore need to convert the vector from a magnitude and direction to component form or vice versa. To do this we will use right triangles and trigonometry.

Going from a Magnitude and Direction to Component Form

To go from a magnitude and direction to component form, we will first draw a right triangle with the hypotenuse being the original vector. The horizontal arm of the triangle will then be the x-component of the vector while the vertical arm is the y component of the vector. If we know the angle of the vector with respect to either the horizontal or the vertical, we can use the sine and cosine relationship to find the x and y components.

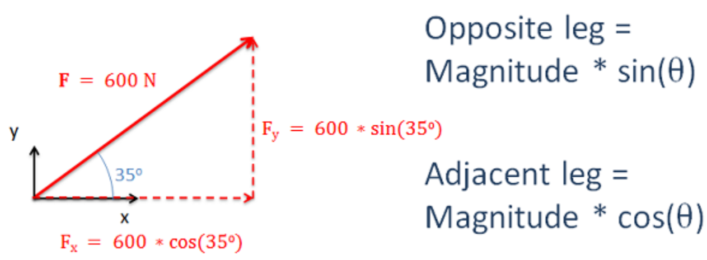


Figure 16.1.3: Using sine and cosine relationships, we can find the x and y components of a vector.

It is important to remember that how we measure the angle will affect the sine and cosine relationships. Multiplying the magnitude by the sine will always give us the opposite leg and multiplying the magnitude by the cosine will always give us the adjacent leg.

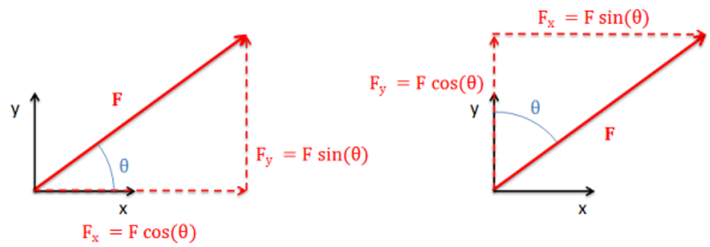


Figure 16.1.4: Depending upon how we measure the angle, the sine/cosine may be either the x or the y component. Remember that the sine will always give you the opposite leg, while the cosine will give you the adjacent leg.

Going from Component Form to Magnitude and Direction

To find the magnitude and the direction of a vector using components, we will use the same process in reverse. We will draw the components as the legs of a right triangle, where the hypotenuse of the triangle shows the magnitude and direction of the vector.

To find the magnitude of the vector we will use the Pythagorean Theorem, taking the square root of the sum of the squares of each component. To find the angle, we can easily use the inverse tangent function, relating the opposite and adjacent legs of our right triangle.

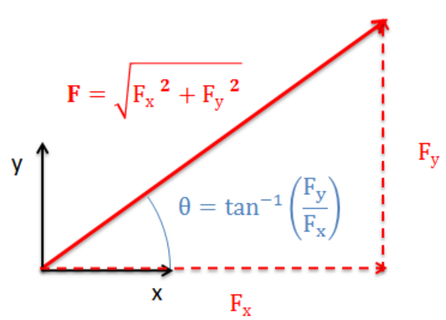


Figure 16.1.5: We will use the Pythagorean Theorem and the inverse tangent function to convert the vector back into magnitude and direction form.

If we know the magnitude of the hypotenuse, we can also use the inverse sine and cosine functions in place of the inverse tangent function to find the angle. As with the previous conversion, it is important to clearly identify the opposite leg, the adjacent leg, and the hypotenuse in our diagrams and to think of these when applying the inverse trig functions.

Converting Between Vector Representations in 3D

In three dimensions, we will have either three components (x, y, and z) for component form or a magnitude and two angles for the direction in magnitude and direction form. To convert between forms we will need to draw in two sets of right triangles. The hypotenuse of the first triangle will be the original vector and one of the legs will be one of the three components. The other leg will then be the hypotenuse of the second triangle. The legs of this second triangle will then be the remaining two components as shown in the diagram below. Use sine and cosine relationships to find the magnitude of each component along the way. This general process of two consecutive right triangles will always hold true, but depending on angles that are given or chosen which components end up being which leg can vary. Carefully plotting everything out in a diagram is important for this reason.

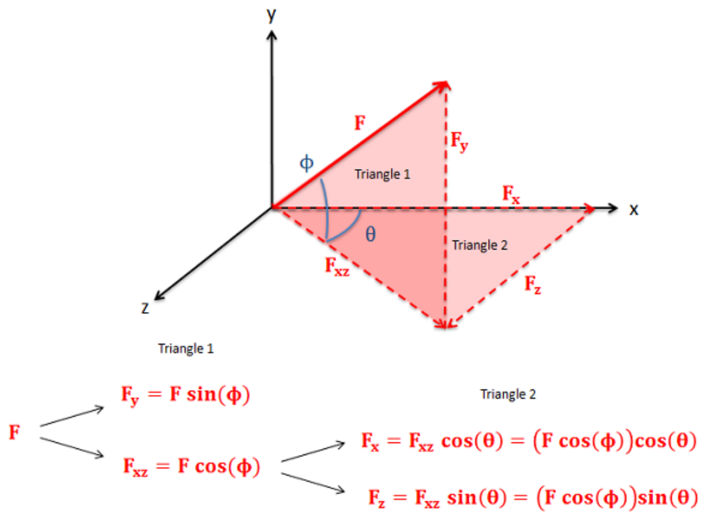


Figure 16.1.6: For 3D vectors we will need to draw two right triangles to convert between forms.

To go from component form back to a magnitude and direction, we will use the 3D form of the Pythagorean Theorem (the magnitude will be the square root of the sum of the three components squared) and we can again use the inverse trig functions to find the angles. We simply need to work backwards through the two right triangles in our problem, so again it is important to carefully draw out your diagrams.

Alternative Method for Finding 3D Vector Components

Sometimes, as with the tension in a cable, the geometry of the cable is given in component form rather than as angles. In cases such as this we could use geometry to figure out the angles and then use those angles to figure out the components, but there is a mathematical shortcut that will allow us to solve for the components more quickly involving the ratio of lengths. Specifically, the ratios of the components of the cable length to the overall length of the cable will be the same as the ratio of the corresponding force components to the overall magnitude of the force.

To use this method we will first need to find the overall length of the cable (or other physical geometric feature) using the Pythagorean Theorem. Once we have that overall length, we find a ratio by taking the x component of the length divided by the overall length. To find the x-component of the force, we simply multiply the overall magnitude of the force by this ratio of lengths (L_x/L). The process for the y and z components follows a similar path, except the ratios would include the y and z component lengths instead of the x component.

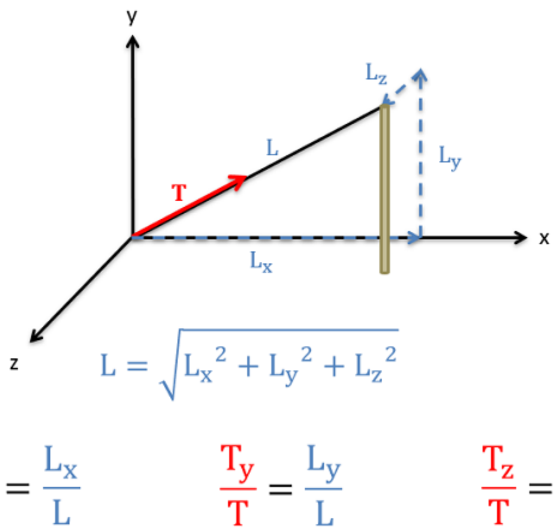


Figure 16.1.7: The alternative method of breaking a vector down into component form relies on the ratios of the cable components to the overall length of the cable.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/2RO7VZ_u0Iw.

Example 16.1.1

Determine the x and y components of the vector shown below.

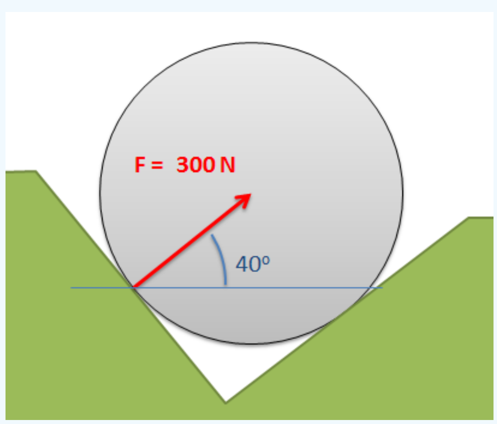


Figure 16.1.8: problem diagram for Example 16.1.1. A force of magnitude 300 N is directed upwards and to the right, at 40° above the horizontal.

Solution:

Video 16.1.2: Worked solution to example problem 16.1.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/TRS6jiih96o.

Example 16.1.2

Determine the x and y components of the vector shown below.

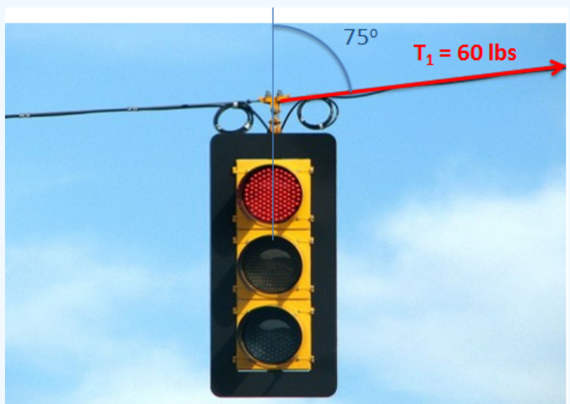


Figure 16.1.9: problem diagram for Example 16.1.2. A tension force of magnitude 60 lbs is exerted along a cable, directed upwards and to the right at a 75° angle from the vertical.

Solution:

Video 16.1.3: Worked solution to example problem 16.1.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/hUrlI6eLGvE.

Example 16.1.3

The velocity vector of the hockey puck shown below is given in component form. Determine the magnitude and direction of the velocity with respect to the axes given.

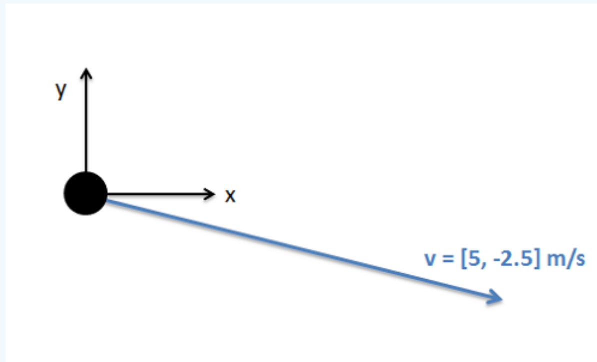


Figure 16.1.10: problem diagram for Example 16.1.3. A hockey puck located at the origin of a Cartesian coordinate plane experiences a velocity of components $[5, -2.5]$ m/s.

Solution:

Video 16.1.4: Worked solution to example problem 16.1.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/3GeIU3Fc_qA.

Example 16.1.4

Determine the x, y, and z components of the force vector shown below.

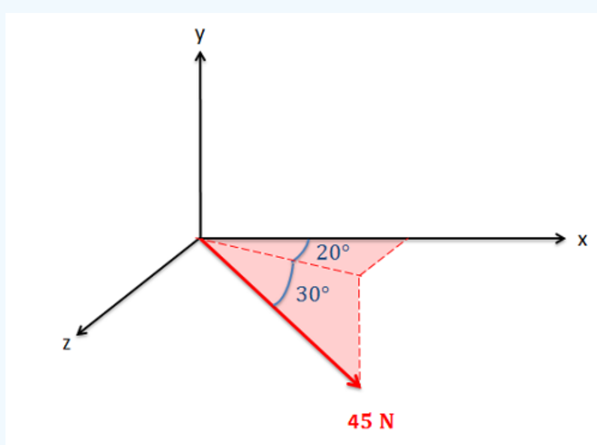


Figure 16.1.11: problem diagram for Example 16.1.4. A 45-Newton force vector with its tail at the origin of a Cartesian coordinate system points 20° out of the screen towards the viewer and 30° below the plane of the horizontal.

Solution:

Video 16.1.5: Worked solution to example problem 16.1.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/ZG31PoIfIEc.

Example 16.1.5

A cable as shown below is used to tether the top of a pole to a point on the ground. The cable has a tension force of 3 kN that acts along the direction of the cable as shown below. What are the x, y, and z components of the tension force acting on the top of the pole?

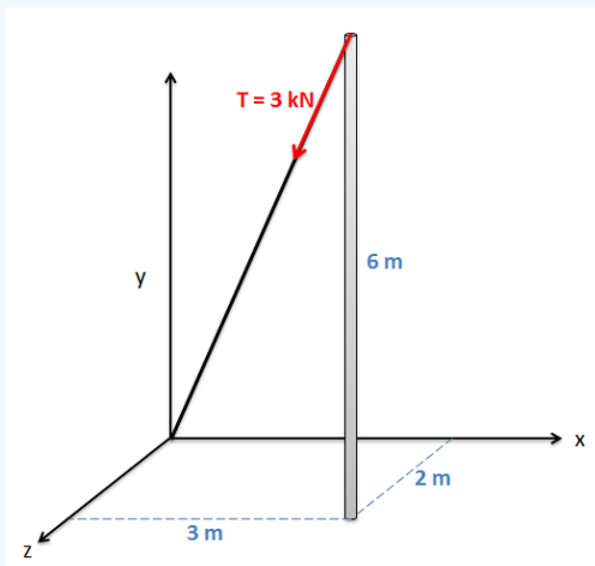


Figure 16.1.12: problem diagram for Example 16.1.5. A cable connects the origin of a Cartesian coordinate system to the top of a 6-meter-long vertical pole with its base at the point (2, 0, 3) meters.

Solution:

Video 16.1.6: Worked solution to example problem 16.1.5, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/dUJwBohfCTU.

16.2: Vector Addition

The most common thing we will need to do with many vector quantities is to add them up. The sum of these vector quantities is the **net** vector quantity. For example, if we have a number of forces acting on a body, the sum of those forces is known as the net force.

The sum of any number of vectors can be determined **geometrically** using the following strategy. Starting with one of the vectors as the base, we redraw the second vector so that the tail of the second vector begins at the tip of the first vector. We can repeat this with a third vector, a forth vector and so on, putting the tail of each vector at the tip of the last vector until we have added taken all vectors into account. Once the vectors are all drawn tip to tail, the sum of all the vectors will be the vector connecting the tail of the first vector to the tip of the last vector.

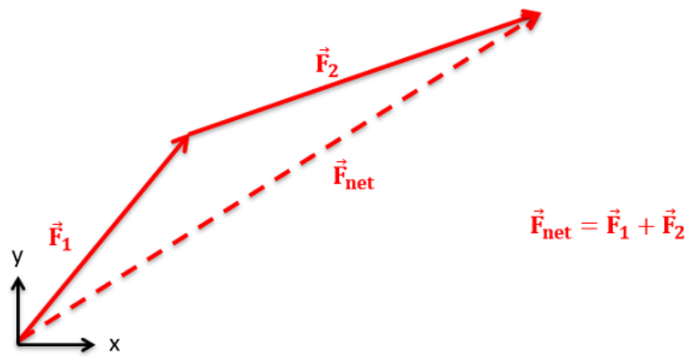


Figure 16.2.1: The geometric addition of vectors involves putting the vectors tip to tail as shown above.

In practice, the easiest way to determine the magnitude and direction of the sum of the vectors is to add the vectors in **component form**. This starts by separating each vector into x, y, and possibly z components. As we can see in the diagram below, the x component of the sum of all the vectors will be the sum of all the x components of the individual vectors. Similarly, the y and z components of the sum of the vectors will be the sum of all the y components and the sum of all the z components respectively.

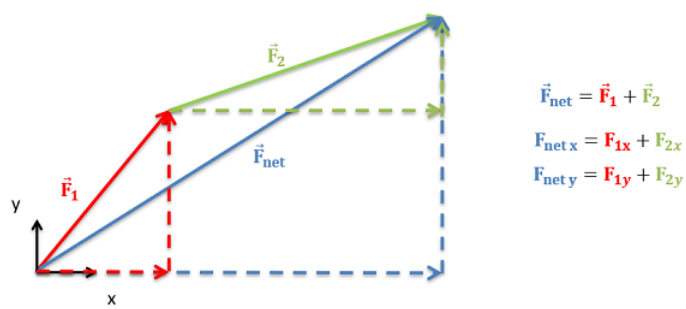


Figure 16.2.2: By summing all the components in a given direction, we can find the component of the sum of the vectors in that direction.

Once we find the sum of the components in each direction, we can either leave the net vector in component form, or we can use the Pythagorean theorem and inverse tangent functions to convert the vector back into a magnitude and direction as detailed on the previous page on vectors.

Figure $PageIndex1$: Vide lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/0tv92MX2_ro.

Example 16.2.1

Determine the sum of the force vectors in the diagram below. Leave the sum in component form.

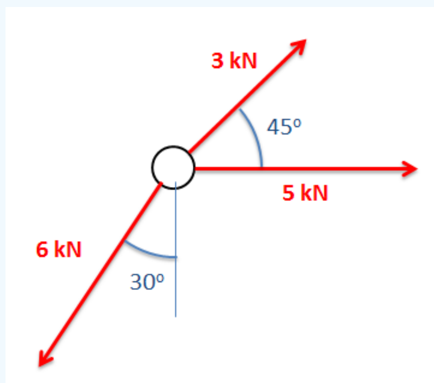


Figure 16.2.3: problem diagram for Example 16.2.1. Three two-dimensional vectors radiate out from a single point.

Solution:

Video 16.2.2: Worked solution to example problem 16.2.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/FwC8ntactEQ.

Example 16.2.2

Determine the sum of the force vectors in the diagram below. Give the sum in terms of a magnitude and a direction.

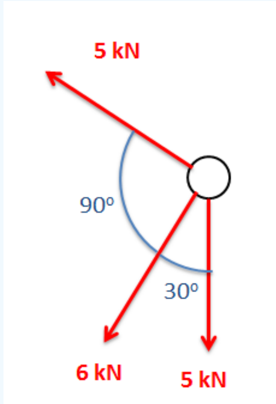


Figure 16.2.4: problem diagram for Example 16.2.2. Three two-dimensional vectors radiate out from a single point.

Solution:

Video 16.2.3: Worked solution to example problem 16.2.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Jj8mCV7rdas.

Example 16.2.3

Determine the sum of the force vectors in the diagram below. Leave the sum in component form.

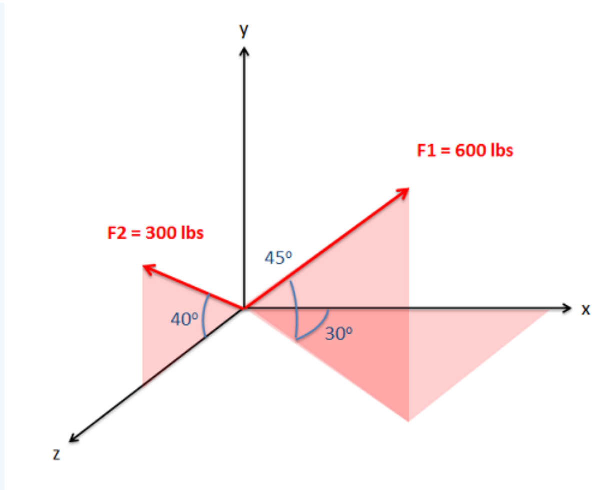


Figure 16.2.5: problem diagram for Example 16.2.3. Two vectors radiate out from the origin of a three-dimensional Cartesian coordinate system.

Solution:

Video 16.2.4: Worked solution to example problem 16.2.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/PZzx3eQp6iQ.

16.3: Dot Product

The **dot product** (also sometimes called the scalar product) is a mathematical operation that can be performed on any two vectors with the same number of elements. The result is a scalar number equal to the magnitude of the first vector, times the magnitude of the second vector, times the cosine of the angle between the two vectors.

$$\vec{A} \cdot \vec{B} = |A||B|\cos(\theta) \tag{16.3.1}$$

In engineering mechanics, the dot product is used almost exclusively with a second vector being a **unit vector**. If the second vector in the dot product operation is a unit vector (thus having a magnitude of 1), the dot product will then represent the magnitude of the first vector in the direction of the unit vector. In this respect, a dot product is useful in determining the component of a given vector in any given direction, where the direction is given in terms of a unit vector.

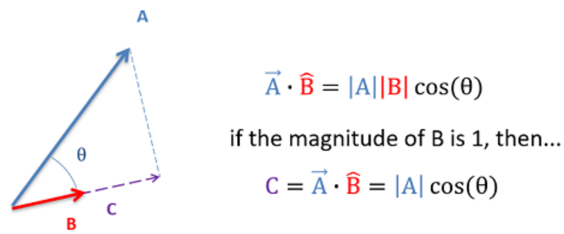


Figure 16.3.1: The dot product of a vector with a unit vector will give you the magnitude of the first vector in the direction of the unit vector.

As an alternative to the above equation for calculating the dot product, we can also calculate the dot product without knowing the angle between the vectors (θ). For this method, we break each vector down into components and take the sum of each set of components multiplied together as shown in the equation below.

$$\vec{A} \cdot \vec{B} = (A_x * B_x) + (A_y * B_y) + (A_z * B_z) \tag{16.3.2}$$

Finally, as with many vector operations, the true strength of the dot product is that computers can calculate them very quickly. Both MATLAB and the Wolfram Alpha Vector Operation Calculator are able to compute dot products for you.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/-MMHU4UhutQ.

Example 16.3.1

Find the dot product of force vector $\vec{A}$ with the given unit vector $\hat{B}$.

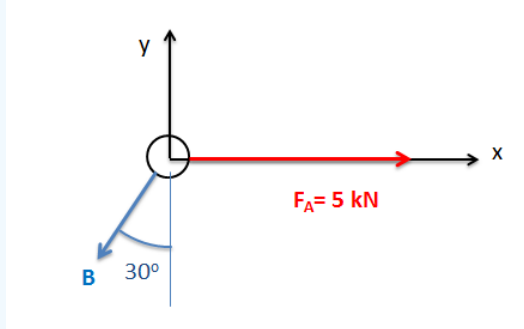


Figure 16.3.2: problem diagram for Example 16.3.1. A force vector and a unit vector radiate out from the origin of a two-dimensional Cartesian coordinate system.

Solution:

Video 16.3.2: Worked solution to example problem 16.3.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/JJ3aXtIZwZ0.

Example 16.3.2

Calculate the dot product of $\vec{A}$ with $\vec{B}$ by hand.

$$\mathbf{A} = [1, 3, 5] \qquad \mathrm{B} = [6, 4, 2]$$

$$\mathbf{A} \cdot \mathbf{B} = ?$$

Figure 16.3.3: problem diagram for Example 16.3.2.

Solution:

Video 16.3.3: Worked solution to example problem 16.3.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/NuuYcDNeja4.

Example 16.3.3

Calculate the dot product of $\vec{A}$ with $\vec{B}$ using MATLAB.

$$\mathbf{A} = [1, 3, 5] \qquad \mathrm{B} = [6, 4, 2]$$

$$\mathbf{A} \cdot \mathbf{B} = ?$$

Figure 16.3.4: problem diagram for Example 16.3.3.

Solution:

Video 16.3.4: Worked solution to example problem 16.3.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/B8PKJtL63C0.

Example 16.3.4

Calculate the dot product of $\vec{A}$ with $\vec{B}$ using the Wolfram Vector Operation Calculator.

$$\mathbf{A} = [1, 3, 5] \qquad \mathrm{B} = [6, 4, 2]$$

$$\mathbf{A} \cdot \mathbf{B} = ?$$

Figure 16.3.5: problem diagram for Example 16.3.4.

Solution:

Video 16.3.5: Worked solution to example problem 16.3.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/2FZSXFASFiQ.

16.4: Cross Product

The cross product is a mathematical operation that can be performed on any two **three-dimensional vectors**. The result of the cross product operation will be a third vector that is **perpendicular** to both of the original vectors and has a magnitude of the first vector times the magnitude of the second vector times the sine of the angle between the vectors.

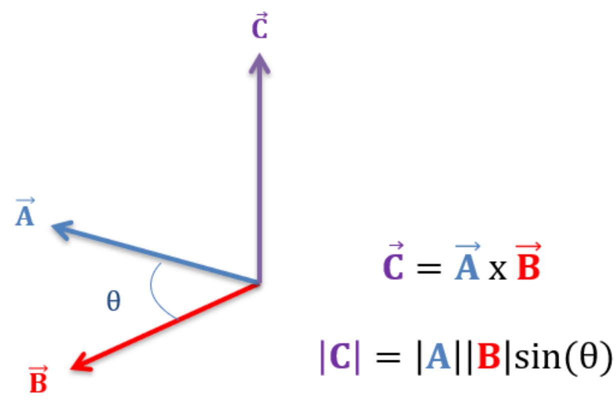


Figure 16.4.1: The cross product of two vectors will be a vector that is perpendicular to both original vectors with a magnitude of $|\vec{A}|$ times $|\vec{B}|$ times the sine of the angle between $\vec{A}$ and $\vec{B}$.

When finding a cross product, you may notice that there are actually two directions that are perpendicular to both of your original vectors. These two directions will be in exact opposite directions. To find which of these two directions the cross product uses, we will use the **right-hand rule**.

To use the right-hand rule, hold out you right hand, point your index finger in the direction of the first vector, turn your middle finger in towards the direction of the second vector, and hold your thumb up. Your thumb should now point in the direction of the cross product vector.

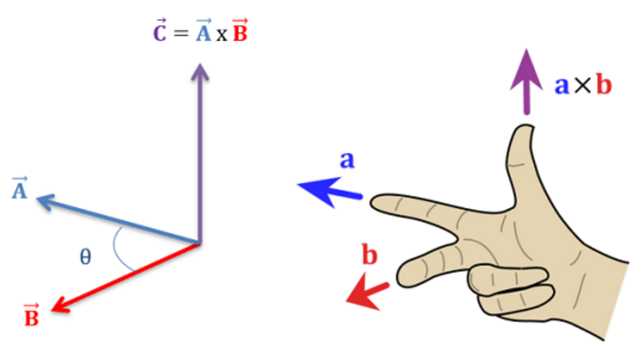


Figure 16.4.2: We can use the right-hand rule to determine the direction of the cross product. Image adapted from work by Acdx, license CC-BY-SA 3.0.

One additional thing you can note with the right-hand rule is that switching the order of the two input vectors (switching $\vec{A}$ and $\vec{B}$) would result in the cross product pointing in exactly the opposite direction. This is because the cross product operation is **not commutative**, meaning that order does matter. Specifically, switching the order of the inputs gives you a result that is exactly the opposite of what your original calculation.

Calculating the Cross Product

To find the cross product by hand, the easiest method is as follows.

1. Write out the letters $x\ y\ z\ x\ y$ in a row as shown in the diagram below.

2. Write out the x, y, and z components of the first vector underneath the corresponding letters of the row of letters from Step 1. Repeat this for the second vector, writing out the second vector's components in a row under the first vector.
3. Draw in diagonals as shown in the diagram. The diagonals that travel to the right as they move down represent positive quantities, while the diagonals that travel to the left as they move down represent negative quantities.
4. Using the letters the diagonals travel through in the top row as a guide for which component of the result each quantity is part of, take the sum of the positive and negative diagonal products for each of the three components in the result. This should give you the final formula shown in the diagram.

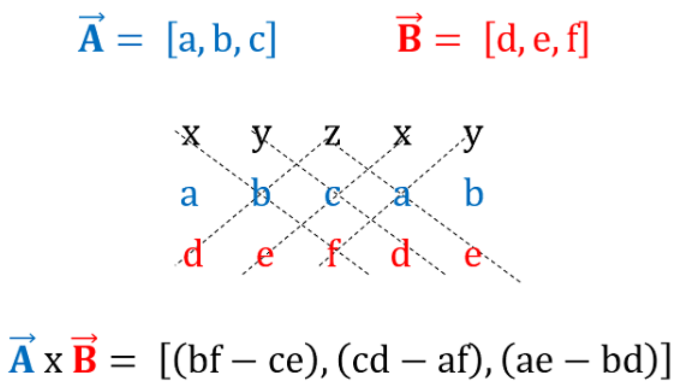


Figure 16.4.3: We can use the visual aid above to help remember the formula for the cross product.

In addition to calculating the cross product by hand, we can also use computer tools such as the "cross" command in MATLAB or web-based tools such as the Wolfram Alpha Vector Operation calculator. Access to these tools allows you to very easily and quickly calculate the cross product and is a major advantage to using vectors operations to analyze problems.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/S0eav05be_U.

Example 16.4.1

Calculate the cross product of force vectors $\vec{A}$ and $\vec{B}$ in the diagram below by hand.

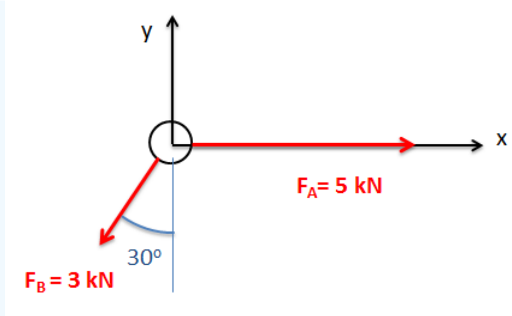


Figure 16.4.4: problem diagram for Example 16.4.1. Two force vectors radiate out from the origin of a Cartesian coordinate plane.

Solution:

Video 16.4.2: Worked solution to example problem 16.4.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/KQhOkEflq9s.

Example 16.4.2

Calculate the cross product of the vectors $\vec{A}$ and $\vec{B}$ in the diagram below by hand.

$$\mathbf{A} = [1, 3, 5] \qquad \mathrm{B} = [6, 4, 2]$$

$$\mathbf{A} \text{ x } \mathbf{B} = ?$$

Figure 16.4.5: problem diagram for Example 16.4.2.

Solution:

Video 16.4.3: Worked solution to example problem 16.4.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/oHLU_q3kVKc.

Example 16.4.3

Calculate the cross product of the vectors $\vec{A}$ and $\vec{B}$ in the diagram below using MATLAB.

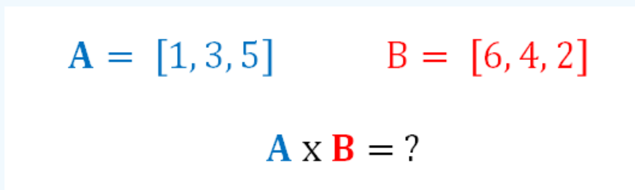


Figure 16.4.6: problem diagram for Example 16.4.3.

Solution:

Video 16.4.4: Worked solution to example problem 16.4.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/a9w6GWQJ96A.

Example 16.4.4

Calculate the cross product of vectors $\vec{A}$ and $\vec{B}$ in the diagram below, using the Wolfram Vector Operation Calculator.

$$\mathbf{A} = [1, 3, 5] \qquad \mathrm{B} = [6, 4, 2]$$

$$\mathbf{A} \text{ x } \mathbf{B} = ?$$

Figure 16.4.7: problem diagram for Example 16.4.4.

Solution:

Video 16.4.5: Worked solution to example problem 16.4.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/24JCFHWWGG4.

16.5: Solving Systems of Equations with Matrices

A **system of equations** is any set of equations that share some variables. A **linear equation** is an equation that consists entirely of constants and simple variables. These variables can only be multiplied by a constant, and cannot be multiplied together, raised to an exponent, used on logs or square roots, or put through other, more complex mathematical functions. An example of a system of linear equations is provided below.

$$F_{AX} + F_{BX} = 0 \tag{16.5.1}$$

$$F_{AY} - 8 = 0 \tag{16.5.2}$$

$$-16 + 4F_{AY} + 8F_{AX} = 0 \tag{16.5.3}$$

In courses such as statics and dynamics, we will often wind up with a system of linear equations and be asked to solve for the unknowns in those equations. When we have just a few equations in our system, we will generally solve the equations by hand using algebraic methods such as substitution or elimination though addition or subtraction. For larger and more complex problems, we can wind up with larger systems of equations and at some point the math may become difficult to handle by hand. For these large systems of linear equations, the easiest way to solve for the unknowns is to convert the system of equations into a single matrix equation, and then use computer tools to solve the matrix equation for unknowns. Some computer tools will let you enter the system of equations manually, but in the background the computer is probably just converting it to a matrix equation in the background. For this reason it can be useful to understand this process.

In terms of assumptions, it is important to mention that this method will only work with systems of **linear equations**, and to have a solvable matrix equation we will need to have the **same number of equations as unknown variables**. For example, above we have a system of equations with three equations and three unknown variables. If these numbers do not match we will not be able to solve the matrix equation using the method described below.

Converting a System of Equations to a Matrix Equation:

The first step in converting a system of equations into a matrix equation is to rearrange the equations into a consistent format. Generally we will put all the variables with their coefficients on one side of the equation and the constants on the other side of the equation. Additionally, it is best to list the variables in the same order in each equation. This process of rearranging the equations will make conversion later on easier.

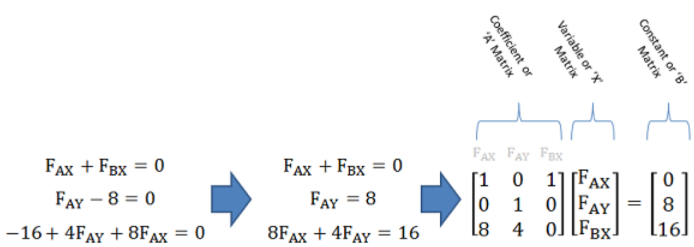


Figure 16.5.1: To convert a system of equations into a single matrix equation, we will first rearrange the equations for a consistent order. Then we will write out the coefficient (A), variable (X), and constant (B) matrices.

Next we will begin the process of writing out the three matrices that make up the matrix equation. These three matrices are the coefficient matrix (often referred to as the A matrix), the variable matrix (often referred to as the X matrix), and the constant matrix (often referred to as the B matrix).

- The **coefficient matrix** (or A matrix) is a $N \times N$ matrix (where N is the number of equations / number of unknown variables) that contains all the coefficients for the variables. Each row of the matrix represents a single equation while each column represents a single variable (it is sometimes helpful to write the variable at the top of each column). For instances where a variable does not show up in an equation, we assume a coefficient of 0.
- The **variable matrix** (or X matrix) is a $N \times 1$ matrix that contains all the unknown variables. It is important that the order of the variables in the coefficient matrix match the order of the variables in the variable matrix.
- Finally, on the other side of the equal sign we have the **constant matrix** (or B matrix). This is a $N \times 1$ matrix containing all the constants from the right side of the equations. It is important that the order of the constants matches the order of equations in the coefficient matrix.

Once we have the three matrices set up, we are ready to solve for the unknowns in the variable matrix.

Solving the Matrix Equation:

Starting with our A, X, and B matrices in the matrix equation below, we are looking to solve for for values of the unknown variables that are contained in our X matrix.

$$[A][X]=[B] \tag{16.5.4}$$

For a scalar equation, we would simply do this by dividing both sides by A, where the value for X would be B/A. With a matrix equation, we will instead need to multiple both sides of the equation by the inverse of the A matrix. This will cancel out the A matrix on the left side, leaving only the X matrix that you are looking for. On the left we will have the inverse of the A matrix times the B matrix. The result of this operation will be a $N\times 1$ matrix containing the solution for all the variables. The value in each row of the solution will correspond to the variable listed in the same row of the X matrix.

$$[X]=[A]^{-1}[B] \tag{16.5.5}$$

It is possible to find the inverse of the A matrix by hand and then multiply this with the B matrix, but this process would take longer than just solving the equations using algebra. The true strength of the method is that computer tools such as MATLAB or Wolfram Alpha can perform the matrix inversion and multiplication for you.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/ZLaeFbqR9QU.

Example 16.5.1

The equilibrium equations for the body shown below are listed on the right. Convert the system of equations into a single matrix equation and solve for the unknowns.

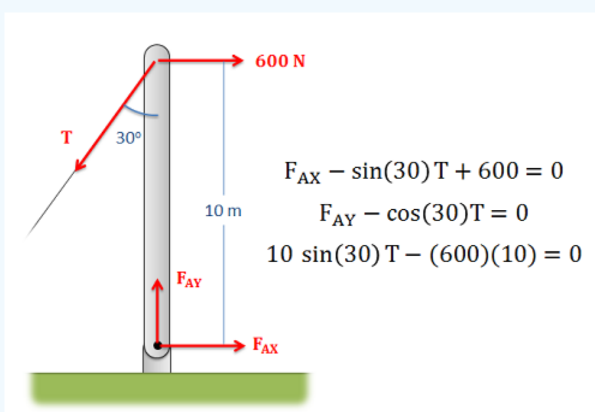


Figure 16.5.2: problem diagram for Example 16.5.1. A system of equations to solve for the forces experienced by a vertical pole attached to the ground.

Solution:

Video 16.5.2: Worked solution to example problem 16.5.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/QX-hJhXTEe0.

16.6: Appendix 1 Homework Problems

Exercise 16.6.1

Determine the x and y components of the force vector shown below.

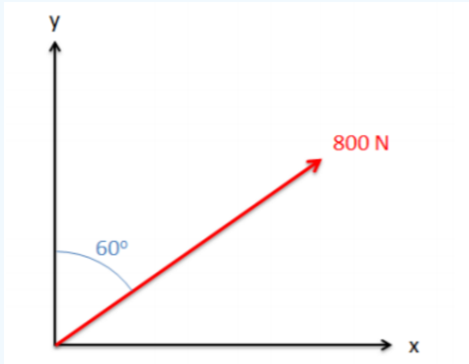


Figure 16.6.1: problem diagram for Exercise 16.6.1. A vector in the first quadrant of a Cartesian coordinate plane, with its tail at the origin.

Solution:

$F_x = 692.8\ N$

$F_y = 400N$

Exercise 16.6.2

Determine the x, y, and z components of the vector shown below.

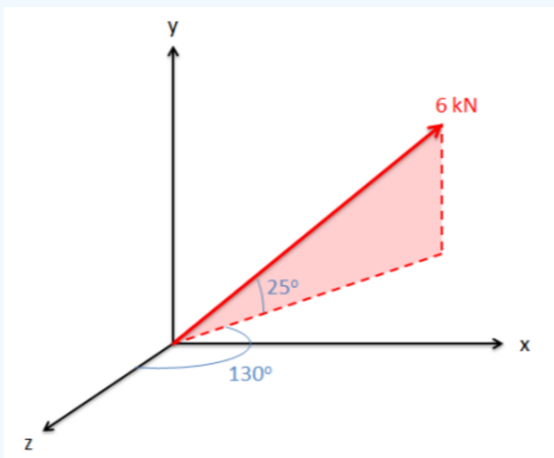


Figure 16.6.2: problem diagram for Exercise 16.6.2. A vector originates from the origin of a Cartesian coordinate system, pointing into the plane of the screen.

Solution:

$F_x = 4.17\ kN$

$F_y = 2.54\ kN$

$F_z = -3.50\ kN$

Exercise 16.6.3

An 80-lb tension acts along a cable stretching from point O to point A. Based on the dimensions given, break the tension force shown into x, y, and z components.

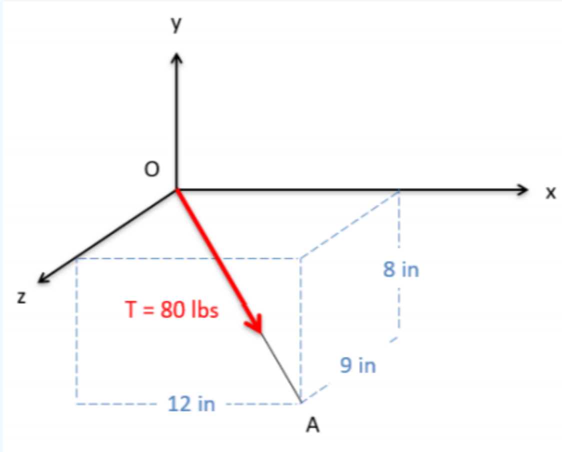


Figure 16.6.3: problem diagram for Exercise 16.6.3. A cable with a force acting along it connects the origin of a Cartesian coordinate system to a given point A.

Solution:

$$F_x = 56.47\ lbs$$

$$F_y = -37.64\ lbs$$

$$F_z = 42.35\ lbs$$

Exercise 16.6.4

Determine the x and y components of the sum of the two vectors shown below.

Figure 16.6.4: problem diagram for Exercise 16.6.4. Two force vectors radiate out from the origin of a Cartesian coordinate plane.

Solution:

$$F_{total} = [58.2, 41.7]\ lbs$$

Exercise 16.6.5

There are two forces acting on a barge as shown below ($\vec{F}_1$ and $\vec{F}_2$). The magnitude and direction of $\vec{F}_1$ is known, but the magnitude and direction of $\vec{F}_2$ is not. If the sum of the two forces is 600 N along the x-axis, what must the magnitude and direction of $\vec{F}_2$ be?

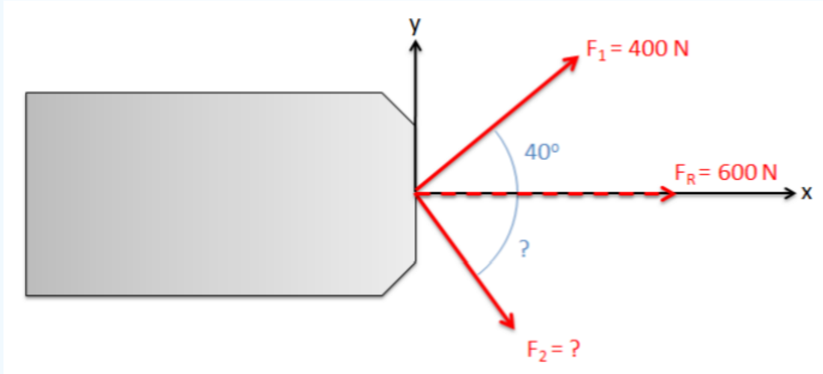


Figure 16.6.5: problem diagram for Exercise 16.6.5. Two forces are applied to the same point on a barge, with the magnitude and direction of one of the force vectors as well as the vector sum being known.

Solution:

$\vec{F}_2 = 390.3\ N$, at 41.2° below the x-axis

CHAPTER OVERVIEW

17: Appendix 2 - Moment Integrals

17.1: Moment Integrals

A **moment integral**, as the name implies, is the general concept of using integration to determine the net moment of a force that is spread over an area or volume. Because moments are generally a force times a distance, and because distributed forces are spread out over a range of distances, we will need to use calculus to to determine the net moment exerted by a distributed force.

$$\int M = \int F(d) * d \tag{17.1.1}$$

Beyond the most literal definition of a moment integral, the term "moment integral" is also general applied the process of integrating distributed **areas** or **masses** that will be resisting some moment about a set axis.

Some of the applications of moment integrals include:

1. Finding point loads that are equivalent to distributed loads (the **equivalent point load**).
2. Finding the **centroid** (geometric center) or **center of mass** for 2D and 3D shapes.
3. Finding the **area moment of inertia** for a beam cross-section, which will be one factor in that beam's resistance to bending.
4. Finding the **polar area moment of inertia** for a shaft cross-section, which will be one factor in that shaft's resistance to torsion.
5. Finding the **mass moment of inertia**, indicating a body's resistance to angular accelerations.

When looking at moment integrals, there are number of different types of moment integrals. These will include moment integrals in one dimension, two dimensions, and three dimensions; moment integrals of force functions, of areas/volumes, or of mass distributions; first order or second order moment integrals; and rectangular or polar moment integrals.

Any combination of these different types is possible (for example, a first-order rectangular 2D area moment integral, or a second-order polar 3D mass moment integral). However, only some of these combinations will have practical applications and will be discussed in detail on future pages.

1D, 2D, and 3D Moment Integrals

Technically we can take the moment integral in any number of dimensions, but for practical purposes we will never deal with moment integrals beyond three dimensions. The number of dimensions will affect the complexity of the calculations (with three-dimensional moment integrals being more involved than one- or two-dimensional moment integrals), but the nature of the problem will dictate the dimensions needed. Often this is not listed in the type of moment integral, requiring you to assume the type based on the context of the problem.

Force, Area/Volume, and Mass Moment Integrals

The next distinction in moment integrals is made with regard what we are integrating. Generally, we can integrate force functions over some distance, area, or volume, we can integrate the area or volume function itself, or we can integrate the mass distribution over the area or volume. Each of these types of moment integrals has a different purpose and will start with a different mathematical function to integrate, but the integration process beyond that will be very similar.

First vs Second Moment Integrals

For moment integrals we will always be multiplying the force function, area or volume function, or mass distribution function by either a distance or a distance squared. First moment integrals just multiply the initial function by the distance, while second moment integrals multiply the function by the distance squared. Again, the type of moment integral we will use depends upon our application, with things like equivalent point load, centroids, and center of mass relying on first moment integrals, and area moments of inertia, polar moments of inertia, and mass moments of inertia relying on second moment integrals. As you can probably deduce from this list, second moment integrals, are often labeled as a **moment of inertia**.

Rectangular vs Polar Moment Integrals

Finally we will talk about **rectangular moment integrals** versus **polar moment integrals**. This is a difference in how we define the distance in our moment integral. Let's start with the distinction in 2D. If our distance is measured from some axis (for example the x-axis, or the y-axis) then it is a rectangular moment integral. On the other hand, if the distance is measured from some point (such as the origin), then it is a polar moment integral.

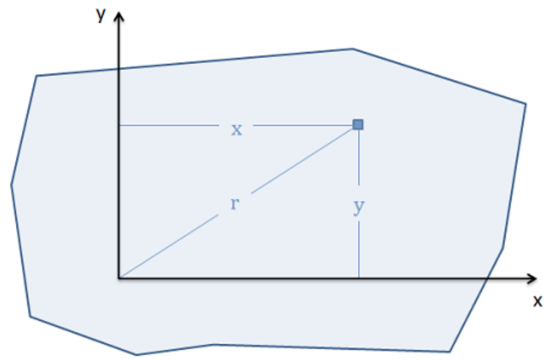


Figure 17.1.1: In two dimensions, if we locate a point by measuring its distance from some axis (similar to what x and y do here) then we have a rectangular moment integral. If we locate it by measuring the distance from some point (as with r here), then we have a polar moment integral.

This distinction is important for how we will take the integral. For rectangular moment integrals we will move left to right or bottom to top. For polar moment integrals we will instead take the integral by radiating out from the center point.

In three-dimensional problems, the definitions change slightly. For rectangular moment integrals, the distance will be measured from some plane (such as the xy-plane, xz-plane, or yz-plane). Again we will integrate left to right, bottom to top, or now back to front with distances corresponding to the x, y or z coordinates of that point. For a polar moment integral, the distance will be measured from some axis (such as the the x, y, or z axis), and we will integrate by radiating outward from that axis.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/kdCCYkWyF40.

17.2: Centroids of Areas via Integration

The **centroid** of an area can be thought of as the geometric center of that area. The location of the centroid is often denoted with a C with the coordinates being $(\bar{x}, \bar{y})$, denoting that they are the average x and y coordinate for the area. If an area was represented as a thin, uniform plate, then the centroid would be the same as the center of mass for this thin plate.

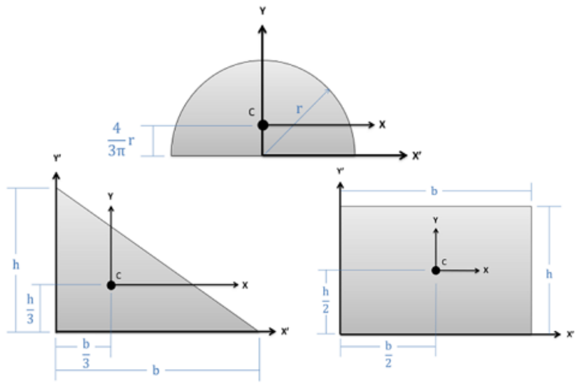


Figure 17.2.1: The centroid (marked C) for a few common shapes.

Centroids of areas are useful for a number of situations in the mechanics course sequence, including in the analysis of distributed forces, the bending in beams, and torsion in shafts, and as an intermediate step in determining moments of inertia.

The location of centroids for a variety of common shapes can simply be looked up in tables, such as this table for 2D centroids and this table for 3D centroids. However, we will often need to determine the centroid of other shapes; to do this, we will generally use one of two methods.

1. We can use the **first moment integral** to determine the centroid location.
2. We can use the **method of composite parts** along with centroid tables to determine the centroid location.

On this page we will only discuss the first method, as the method of composite parts is discussed in a later section. The tables used in the method of composite parts, however, are derived via the first moment integral, so both methods ultimately rely on first moment integrals.

Finding the Centroid via the First Moment Integral

When we find the centroid of a two-dimensional shape, we will be looking for both an x and a y coordinate, represented as $\bar{x}$ and $\bar{y}$ respectively. Collectively, this $(\bar{x}, \bar{y}$ coordinate is the centroid of the shape.

To find the average x-coordinate of a shape ($\bar{x}$), we will essentially break the shape into a large number of very small and equally sized areas, and find the average x-coordinate of these areas. To do this sum of an infinite number of very small things, we will use integration. Specifically, we will take the **first rectangular area moment integral** along the x-axis, and then divide that integral by the total area to find the average coordinate. We can do something similar along the y-axis to find our $\bar{y}$ value. Writing all of this out, we have the equations below.

$$C = (\bar{x}, \bar{y}) \tag{17.2.1}$$

$$\bar{x} = \frac{\int_A (dA * x)}{A} \tag{17.2.2}$$

$$\bar{y} = \frac{\int_A (dA * y)}{A} \tag{17.2.3}$$

Next let's discuss what the variable dA represents and how we integrate it over the area. The variable dA is the rate of change in area as we move in a particular direction. For $\bar{x}$ we will be moving along the x-axis, and for $\bar{y}$ we will be moving along the y-axis in these integrals.

As we move along the x-axis of a shape from its leftmost point to its rightmost point, the rate of change of the area at any instant in time will be equal to the height of the shape that point times the rate at which we are moving along the axis (dx). Because the height of the shape will change with position, we do not use any one value, but instead must come up with an equation that describes the height at any given value of x. We will then multiply this dA equation by the variable x (to make it a moment integral), and integrate that equation from the leftmost x position of the shape (x_{min}) to the rightmost x position of the shape (x_{max}).

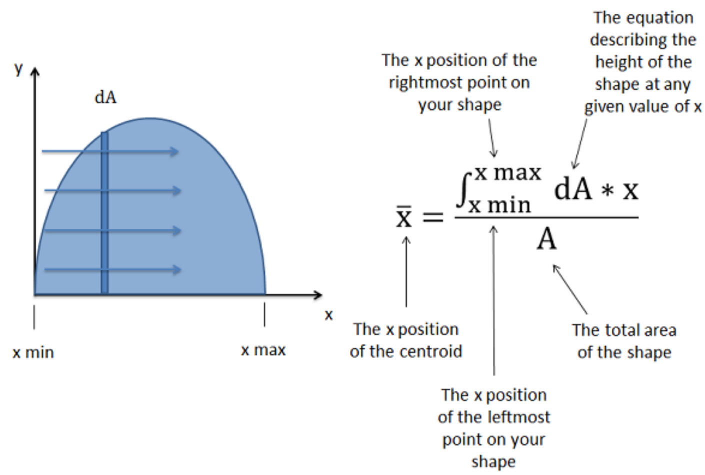


Figure 17.2.2: The procedure for calculating the x coordinate of the centroid.

To find the y coordinate of the of the centroid, we have a similar process, but because we are moving along the y-axis, the value dA is the equation describing the width of the shape times the rate at which we are moving along the y axis (dy). We then take this dA equation and multiply it by y to make it a moment integral. We will integrate this equation from the y position of the bottommost point on the shape (y_{min}) to the y position of the topmost point on the shape (y_{max}).

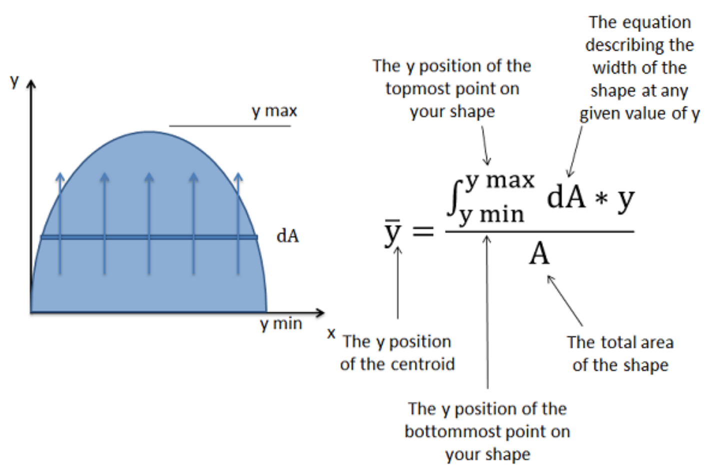


Figure 17.2.3: The procedure for calculating the y coordinate of the centroid.

Using the first moment integral and the equations shown above, we can theoretically find the centroid of any shape as long as we can write out equations to describe the height and width at any x or y value respectively. For more complex shapes, however, determining these equations and then integrating these equations can become very time-consuming. That is why most of the time, engineers will instead use the method of composite parts or computer tools.

Using Symmetry as a Shortcut

Shape symmetry can provide a shortcut in many centroid calculations. Remember that the centroid is located at the average x and y coordinate for all the points in the shape. If the shape has a line of symmetry, that means each point on one side of the line must have an equivalent point on the other side of the line. This means that the average value (AKA the centroid) must lie along any axis of symmetry. If the shape has more than one axis of symmetry, then the centroid must exist at the intersection of the two axes of symmetry.

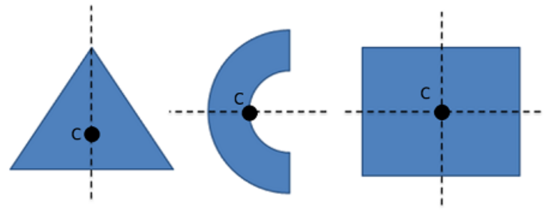


Figure 17.2.4: If a shape has a line of symmetry, the centroid must lie somewhere along that line. If a shape has more than one line of symmetry, the centroid must exist at the intersection of these lines.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/ilgYN8c1HUk.

Example 17.2.1

Find the x and y coordinates of the centroid of the shape shown below.

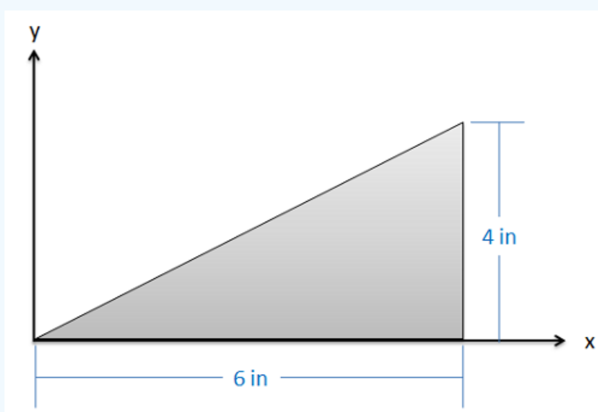


Figure 17.2.5: problem diagram for Example 17.2.1. A right triangle with one leg lying along the x-axis in the first quadrant of a Cartesian plane.

Solution:

Video 17.2.2: Worked solution to example problem 17.2.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/dZ2O3zwvuP0.

Example 17.2.2

Find the x and y coordinates of the centroid of the shape shown below.

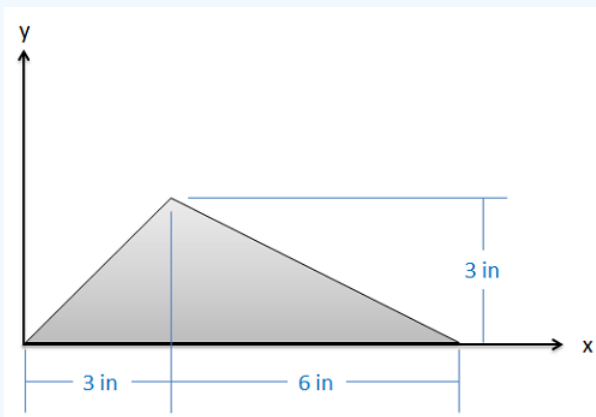


Figure 17.2.6: problem diagram for Example 17.2.2. A triangle with its longest side lying along the x-axis in the first quadrant of a Cartesian plane.

Solution:

Video 17.2.3: Worked solution to example problem 17.2.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/hmGiCFwgIo8.

Example 17.2.3

Find the x and y coordinates of the centroid of the shape shown below.

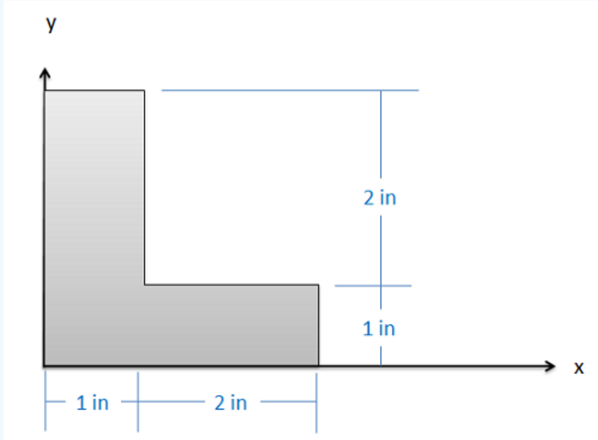


Figure 17.2.7: problem diagram for Example 17.2.3. An L-shaped region lies along the x- and y-axes in the first quadrant of a Cartesian plane.

Solution:

Video 17.2.4: Worked solution to example problem 17.2.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/4MHflda4FBw.

17.3: Centroids in Volumes and Center of Mass via Integration

The **centroid** of a volume can be thought of as the geometric center of that shape. It is often denoted as C, being being located at the coordinates $(\bar{x}, \bar{y}, \bar{z})$. If this volume represents a part with a uniform density (like most single material parts) then the centroid will also be the center of mass, a point usually labeled as G.

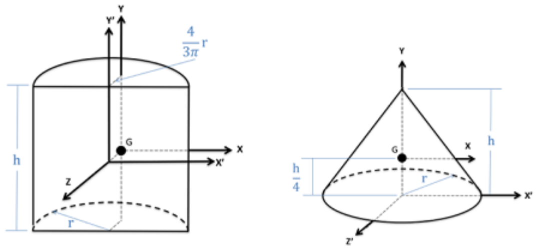


Figure 17.3.1: The centroid point (C) or the center of mass (G) for some common shapes. (The centroid and center of mass are the same point for bodies with a uniform density)

Just as with the centroids of an area, centroids of volumes and the center of mass are useful for a number of situations in the mechanics course sequence, including the analysis of distributed forces, simplifying the analysis of gravity (which is itself a distributed force), and as an intermediate step in determining mass moments of inertia.

Just as with areas, the location of the centroid (or center of mass) for a variety of common shapes can simply be looked up in tables, such as this one. However, we will often need to determine the centroid or center of mass for other shapes, and to do this we will generally use one of two methods.

1. We can use the **first moment integral** to determine the centroid or center of mass location.
2. We can use the **method of composite parts** along with centroid tables to determine the centroid or center of mass location.

On this page we will only discuss the first method, as the method of composite parts is discussed in a later section. However, the tables used in the method of composite parts are derived via the first moment integral, so both methods ultimately rely on first moment integrals.

Finding the Centroid of a Volume via the First Moment Integral

When we find the centroid of a three-dimensional shape, we will be looking for the x, y, and z coordinates ($\bar{x}$, $\bar{y}$, and $\bar{z}$) of the point that is the centroid of the shape.

Much like the centroid calculations we did with two-dimensional shapes, we are looking to find the shape's average coordinate in each dimension. We do this by summing up all the little bits of volume times the x, y, or z coordinate of that bit of volume and then dividing that sum by the total volume of the shape. Again we will use calculus to sum up an infinite number of infinitely small volumes. Specifically this sum will be the **first rectangular volume moment integral** for the shape.

$$C = G = (\bar{x}, \bar{y}, \bar{z}) \quad \text{for constant-density shapes}$$

Working in each of the three coordinate directions we wind up with the following three equations.

$$\bar{x} = \frac{\int_V (dV * x)}{V}$$

$$\bar{y} = \frac{\int_V (dV * y)}{V}$$

$$\bar{z} = \frac{\int_V (dV * z)}{V}$$

With these new equations we have the variable dV rather than dA, because we are integrating over a volume rather than an area. This represents the rate of change of the volume as we move along an axis from one end to another. The rate of change of the volume at any point on the shape will be the **cross-sectional area** that is perpendicular to that axis times the rate at which we are moving along that axis. Since cross-sectional area may vary as we move along the axis, we will need to determine a formula for the cross sectional area at any point along that axis.

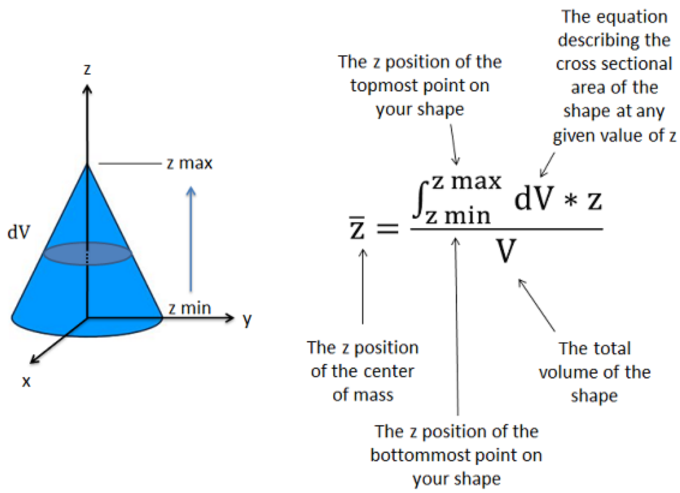


Figure 17.3.2: The z-coordinate of the centroid/center of mass.

Using the first moment integral and the equations shown above, we can theoretically find the centroid of any volume as long as we can write an equation to describe the cross section area for each direction. For more complex shapes, however, determining these equations and then integrating these equations may become very time-consuming. For these complex shapes, the method of composite parts or computer tools will most likely be much faster.

Using Symmetry as a Shortcut

Just as with 2D areas, shape symmetry can provide a shortcut in many centroid calculations. Remember that the centroid coordinate is the average x, y, and z coordinate for all the points in the shape. If the volume has a plane of symmetry, that means each point on one side of the line must have an equivalent point on the other side of the line. This means that the average value (AKA the centroid) must lie within that plane. If the volume has more than one plane of symmetry, then the centroid must exist at the intersection of those planes.

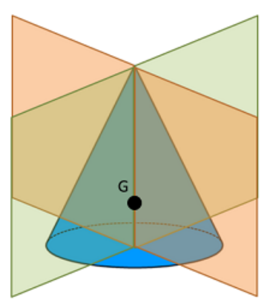


Figure 17.3.3: If a volume has a plane of symmetry, then the centroid must lie somewhere in that plane. If the volume has more than one plane of symmetry, the centroid must exist at the intersection of those planes.

Finding the Center of Mass for Non-Uniform Density Shapes

If a body has a non-uniform density, then the centroid of the volume and the center of mass will no longer be the same point. In these cases, we will be integrating with respect to mass, rather than integrating the volume. To do this, we will use something called a density function, which will provide the density (the mass per unit volume) in terms of the x, y, and/or z location. In the end, we will also be dividing by total mass, rather than dividing by total volume. The generalized equations for the center of mass are shown in the equations below.

$$C \neq G \tag{17.3.1}$$

$$G = (x_G, y_G, z_G) \tag{17.3.2}$$

$$x_G = \frac{\int_m (dm * x)}{m} = \frac{\int_V (\rho(x,y,z) * dV * x)}{m} \tag{17.3.3}$$

$$y_G = \frac{\int_m (dm * y)}{m} = \frac{\int_V (\rho(x,y,z) * dV * y)}{m} \tag{17.3.4}$$

$$z_G = \frac{\int_m (dm * z)}{m} = \frac{\int_V (\rho(x,y,z) * dV * z)}{m} \tag{17.3.5}$$

In instances of uniform density (where the density function did not vary with location and was therefore just a constant), the density constant could be moved outside of the integral. On the bottom, we could also write mass as density times volume, and the density terms on the top and bottom of the fraction would cancel out. That is why for uniform-density parts, the centroid and center of mass will be the same point.

When we have a density function that is not a constant, we will have to come up with a mathematical function for the density in terms of x and/or y and/or z locations. If density varies along more than one axis, determining the function and then integrating it may become quite difficult, and computer modeling may be advisable in these situations.

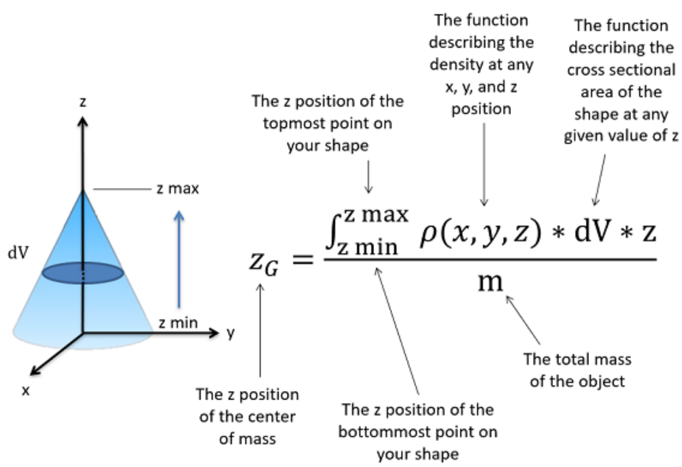


Figure 17.3.4: To find the center of mass of a body with a continuously varying density, we must have an equation to describe the density based on position.

Once you have the density function, you will multiply that by the relevant dV function as discussed earlier on the page and multiply it by the variable for the relevant axis. This entire function is integrated from left to right, bottom to top, or back to front, and then that quantity is divided by mass to find the location of the center of mass.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/cBiWKfBLEhk.

Example 17.3.1

The cone shown below is four inches tall and has a four-inch-diameter base. Find the x, y, and z coordinates of the centroid.

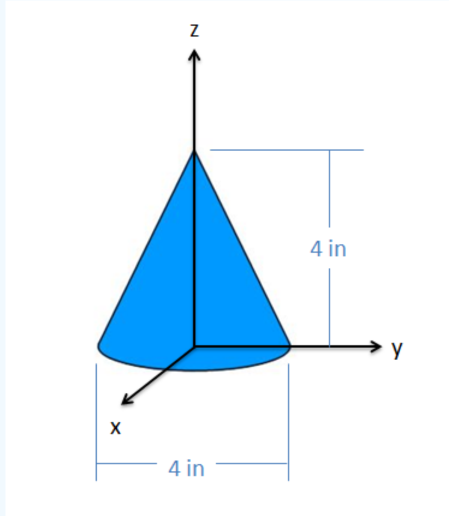


Figure 17.3.5: problem diagram for Example 17.3.1. A cone has a circular base centered on the xy-plane of a Cartesian coordinate system, and a central axis that stretches along the positive z-axis.

Solution:

Video 17.3.2: Worked solution to example problem 17.3.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/IJHwNaOG4-s.

Example 17.3.2

Find the y coordinate of the centroid for the tetrahedron shown in the image below. (The fourth vertex is at the origin)

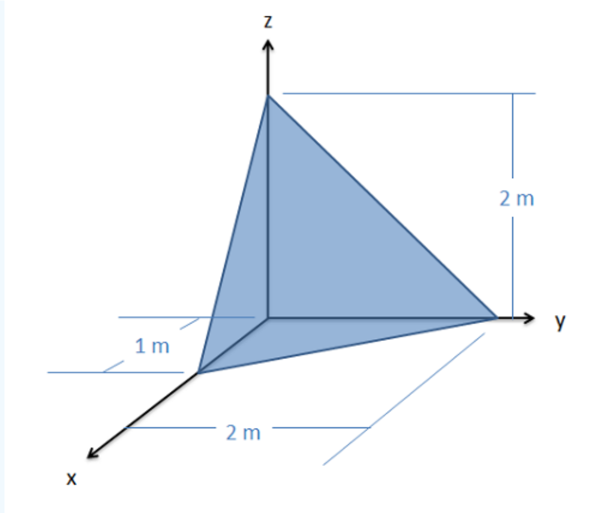


Figure 17.3.6: problem diagram for Example 17.3.2. A tetrahedron lies in the first octant of a Cartesian coordinate system, with three of its edges formed by the coordinate axes.

Solution:

Video 17.3.3: Worked solution to example problem 17.3.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/itu_P4ysw9g.

Example 17.3.3

A water and ceramic slurry with a uniform density of 1100 kilograms per meter cubed enters a settling tank with a height of 1 meter and a diameter of 1 meter. After one hour in the tank, the density of the slurry at the top of the tank is measured to be 1000 kilograms per meter cubed and the density at the bottom of the tank is measured to be 1200 kilograms per meter cubed. Assume the density of the slurry varies linearly between the top and the bottom. How far has the center of mass of the slurry dropped between the initial conditions and the current state?

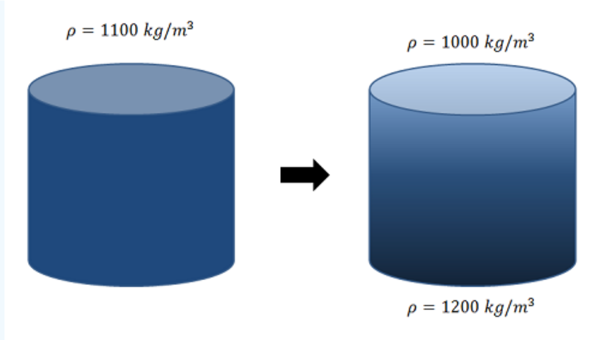


Figure 17.3.7: problem diagram for Example 17.3.3. As particles in a tank settle, a solution of originally uniform density becomes one with linearly decreasing density from bottom to top.

Solution:

Video 17.3.4: Worked solution to example problem 17.3.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/BTyO2HolIHY.

17.4: Centroids and Centers of Mass via Method of Composite Parts

As an alternative to the use of moment integrals, we can use the **Method of Composite Parts** to find the **centroid** of an area or volume or the **center of mass** of a body. This method is is often easier and faster that the integration method; however, it will be limited by the table of centroids you have available. The method works by breaking the shape or volume down into a number of more basic shapes, identifying the centroids or centers of masses of each part via a table of values, and then combing the results to find the overall centroid or center of mass.

A key aspect of the method is the use of these centroid tables. This is a set of tables that lists the centroids (and usually also moments of inertia) for a number of common areas and/or volumes. Some centroid tables can be found here for 2D shapes, and here for 3D shapes. The method of composite parts is limited in that we will need to be able to break our complex shape down entirely into shapes found in the centroid table we have available; otherwise, the method will not work without us also doing some moment integrals.

Finding the Centroid via the Method of Composite Parts

Start the process by labeling an origin point and axes on your shape. It will be important to measure all locations from the same point. Next, we must break our complex shape down into several simpler shapes. This may include areas or volumes (which we will count as positive areas or volumes) or holes (which we will count as negative areas or volumes). Each of these shapes will have a centroid (C) or center of mass (G) listed on the diagram.

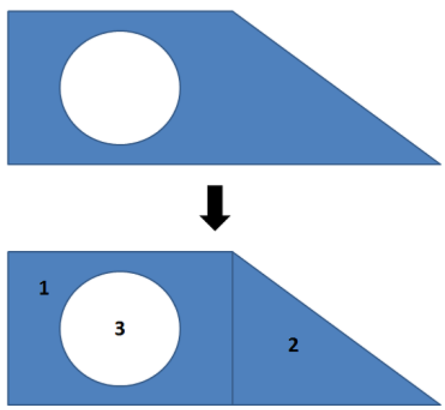
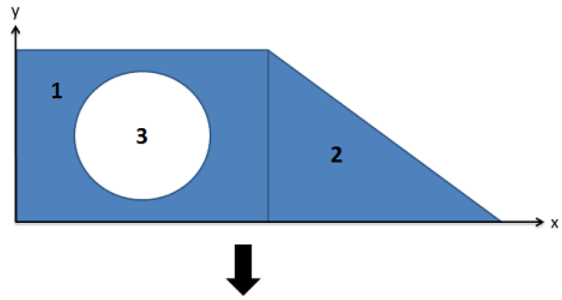


Figure 17.4.1: For the shape shown at the top, we can break it down into a rectangle (1), a right triangle (2), and a circular hole (3). Each of these simple shapes is something we have listed in the centroid table to the right.

Once we have identified the different parts, we will create a table listing the area or volume of each piece, and the x and y centroid coordinates (or x, y, and z coordinates in 3D). It is important to remember that each coordinate you list should be **relative to the same base origin point** that you drew in earlier. You may need to mentally adjust diagrams in the centroid tables so that the shape is oriented in the right direction, and account for the placement of the shape relative to the axes in your diagram.



Shape	Area	$\bar{X}$	$\bar{Y}$
1			
2			
3			

Figure 17.4.2: For each of the shapes, we need to find the area and the x and y coordinates of the centroid. Remember to find the centroid coordinates relative to a single set of axes that is the same for all the shapes.

Once you have the areas and centroid coordinates for each shape relative to your origin point, you can find the x and y coordinate of the centroid for the overall shape with the following formulas. Remember that areas or volumes for any shape that is a hole or cutout in the design will be a negative area in your formula.

$$\bar{x}_{total} = \frac{\sum A_i \bar{x}_i}{A_{total}} \qquad \bar{y}_{total} = \frac{\sum A_i \bar{y}_i}{A_{total}} \tag{17.4.1}$$

This generalized formula to find the centroid's x-location is simply Area 1 times $\bar{x}_1$, plus Area 2 times $\bar{x}_2$, plus Area 3 times $\bar{x}_3$, adding up as many shapes as you have in this fashion and then dividing by the overall area of your combined shape. The equations are the same for the y-location of the overall centroid, except you will instead be using $\bar{y}$ values in your equations.

For centroids in three dimensions we will simply use volumes in place of areas, and we will have a z coordinate for our centroid as well as the x and y coordinates.

Finding the Center of Mass via the Method of Composite Parts

To use the method of composite parts to find the center of mass, we simply need to adjust the process slightly. First, center of mass calculations will always be in three dimensions. Draw an origin point and some axes on your diagram we did for the centroid. We will measure all locations relative to this origin point. We will then need to break the complex shape down into simple volumes, with each simple volume being something in the centroid table we have available. Remember that when we have a part with a uniform material, the centroid and center of mass are the same point, so we will often talk about these interchangeably.

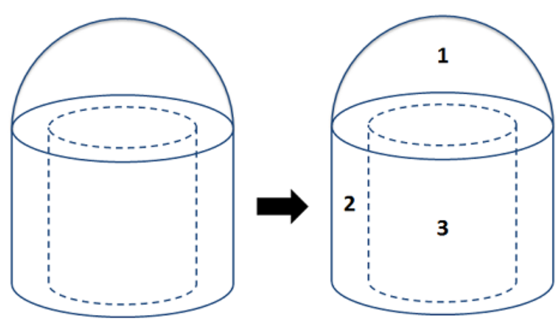
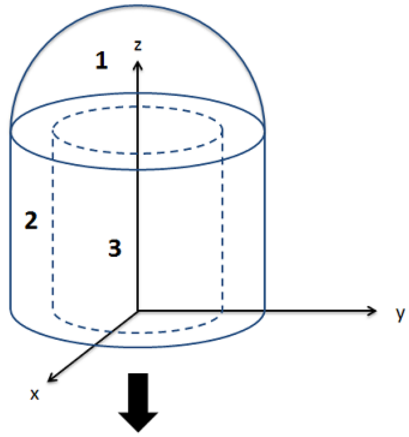


Figure 17.4.3: When finding the center of mass via composite parts, we will break the shape up into several simpler shapes. The figure on the left can be thought of as a hemisphere (1), on top of a cylinder (2) with another smaller cylinder cut out of it (3). Each of these simple volumes are listed in our centroid table.

Once we have identified the different parts, we will create a table indicating the **mass** of each part, and the x, y, and z coordinate of the center of mass for each individual part. It is important to remember that each coordinate you list should be relative to the same base origin point, so you will need to mentally rotate and position the parts in the table on your axes.



Shape	Mass	$\bar{X}$	$\bar{Y}$	$\bar{Z}$
1				
2				
3				

Figure 17.4.4: Create a table with the mass of each piece of the total shape, as well as the center of mass location (x, y, and z coordinates) for each piece.

One complicating factor with mass can be measuring the mass of the pieces separately. If we have a scale, we may simply know the overall mass without knowing the mass of the individual pieces. In these cases, you may need work backwards to calculate the density of the material (by dividing the overall mass by overall volume), and then use density times piece volume to find the mass of each piece individually. When doing this, remember to count cutouts as negative mass in your calculations. For example, for the hollow cylinder in the shape above, you would find the mass of a solid cylinder for Shape 2, then have a negative mass for the cylindrical cutout for Shape 3.

Finally, once you have the mass the and center of mass coordinates for each shape, you can find the coordinates of the center of mass for the overall volume with the following formulas.

$$x_G = \frac{\sum m_i \bar{x}_i}{m_{total}} \qquad y_G = \frac{\sum m_i \bar{y}_i}{m_{total}} \qquad z_G = \frac{\sum m_i \bar{z}_i}{m_{total}} \tag{17.4.2}$$

Similar to the centroid equations, the x-equation is simply the mass of Shape 1 times $\bar{x}_1$, plus the mass of Shape 2 times $\bar{x}_2$, and so on for each part. After you have summed up these products for all the shapes, just divide by the total mass.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/wfjLNSfPXAI.

Example 17.4.1

Find the x and y coordinates of the centroid of the shape shown below.

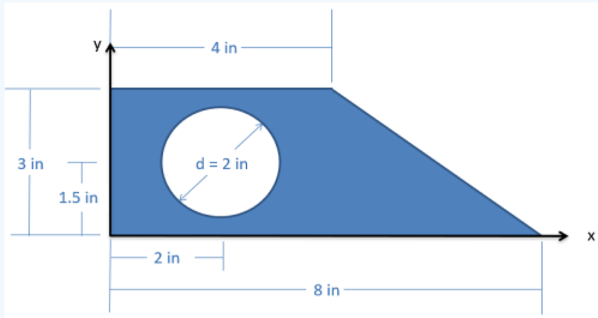


Figure 17.4.5: problem diagram for Example 17.4.1. A trapezoid with two perpendicular sides and a hole through its middle lies along the axes of the first quadrant of a Cartesian coordinate plane.

Solution:

Video 17.4.2: Worked solution to example problem 17.4.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/QIe6Hk4Bofs.

Example 17.4.2

Find the x and y coordinates of the centroid of the shape shown below.

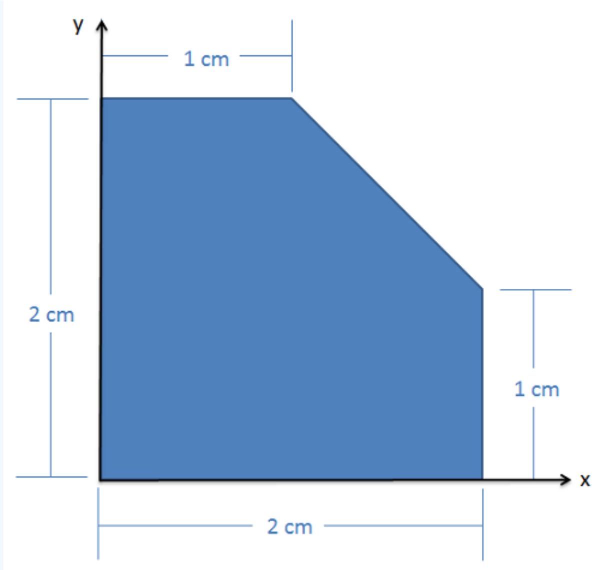


Figure 17.4.6: problem diagram for Example 17.4.2. A pentagon with two perpendicular sides lies along the axes of the first quadrant of a Cartesian coordinate plane.

Solution:

Video 17.4.3: Worked solution to example problem 17.4.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/F1rlzboPlZM.

Example 17.4.3

Find the x and y coordinates of the centroid of the shape shown below.

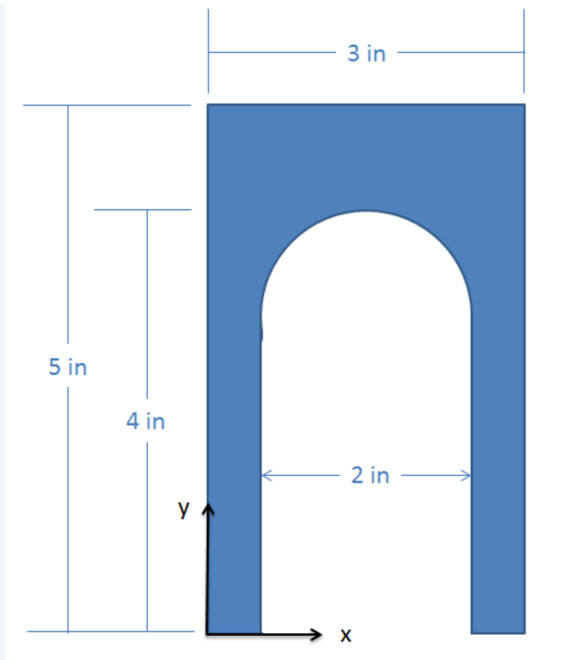


Figure 17.4.7: problem diagram for Example 17.4.3. A rectangle with a hole through one side (shaped like a rectangle topped with a semicircle) lies along the axes of the first quadrant of a Cartesian coordinate plane.

Solution:

Video 17.4.4: Worked solution to example problem 17.4.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/tLybTEX8S_I.

Example 17.4.4

The shape shown below consists of a solid semicircular hemisphere on top of a hollow cylinder. Based on the dimensions below, determine the location of the centroid.

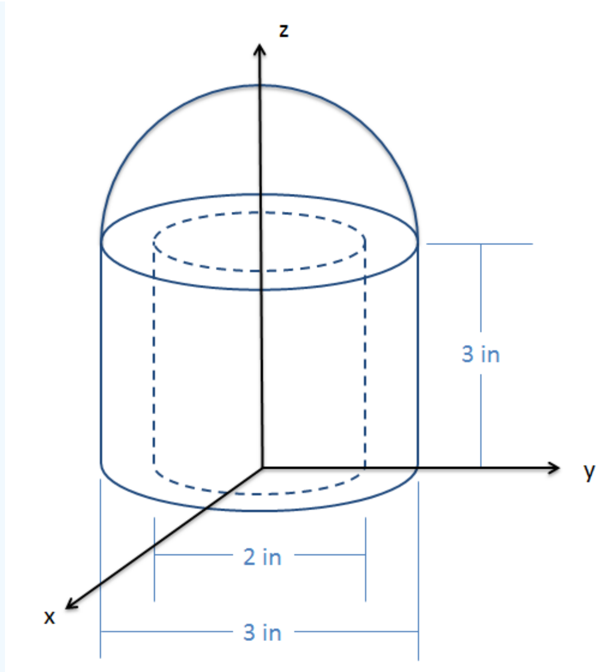


Figure 17.4.8: problem diagram for Example 17.4.4. A hollowed-out cylinder topped with a hemisphere lies along the z-axis of a Cartesian coordinate system, with its base being centered in the xy-plane.

Solution:

Video 17.4.5: Worked solution to example problem 17.4.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/vQk4OqTcDpQ.

Example 17.4.5

A spherical steel tank (density = 8050 kg/m^3) is filled halfway with water (density = 1000 kg/m^3) as shown below. Find the overall mass of the tank and the current location of the center of mass of the tank (measured from the base of the tank).

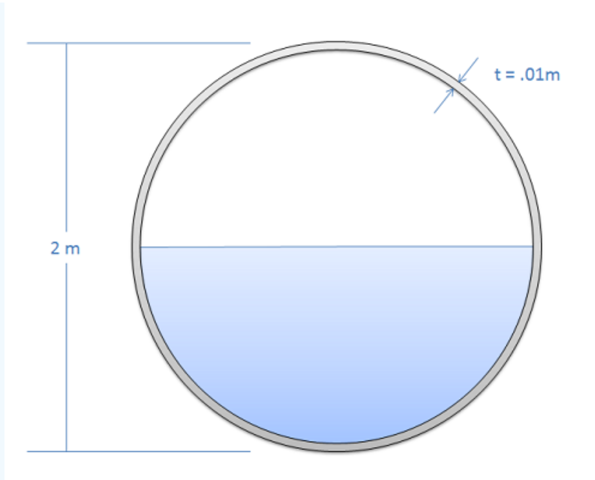


Figure 17.4.9: problem diagram for Example 17.4.5. A spherical steel tank of diameter 2 meters and metal thickness 0.01 meters is half-filled with water.

Solution:

Video 17.4.6: Worked solution to example problem 17.4.5, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/5zbYD4Wogck.

17.5: Area Moments of Inertia via Integration

Area moments of inertia are used in engineering mechanics courses to determine a body's resistance to bending loads or torsional loads. Specifically, the area moment of inertia refers to the **second area moment integral** of a shape, with I_{xx} representing the moment of inertia about the x-axis, I_{yy} representing the moment of inertia about the y-axis, and J_{zz} (also called the polar moment of inertia) representing the moment of inertia about the z-axis. The moment of inertia about each axis represents the shapes resistance to a moment applied about that respective axis. Moments about the x- and y-axes would tend to bend an object, while moments about the z-axis would tend to twist the body.

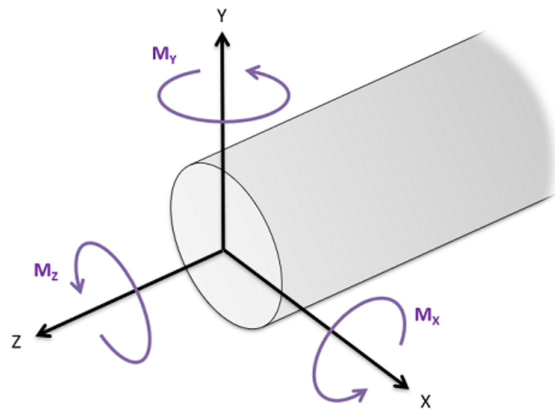


Figure 17.5.1: The moments of inertia for the cross section of a shape about each axis represents the shape's resistance to moments about that axis. Moments applied about the x-axis and y-axis represent bending moments, while moments about the z- axis represent torsional moments.

Just as with centroids, each of these moments of inertia can be calculated via **integration** or by using the method of **composite parts** and the **parallel axis theorem**. On this page we are going to focus on calculating the area moments of inertia via moment integrals.

Bending Stresses and the Second Area Moment

When an object is subjected to a bending moment, that body will experience both internal tensile stresses and compressive stresses as shown in the diagram below. These stresses exert a net moment to counteract the loading moment, but exert no net force so that the body remains in equilibrium.

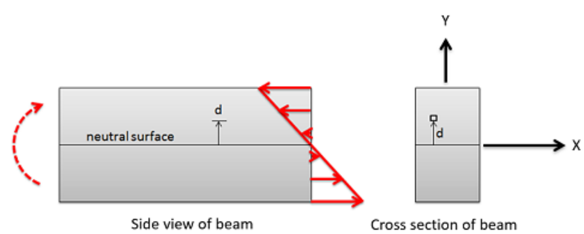


Figure 17.5.2: A bending moment and the resulting internal tensile and compressive stresses needed to ensure the beam is in equilibrium.

As we can see in the diagram, there is some central plane along which there are no tensile or compressive stresses. This is known as the neutral surface, and if there are no other forces present it will run through the centroid of the cross section. As we move up or down from the neutral surface, the stresses increase linearly. The moment exerted by this stress at any point will be the stress times the moment arm, which also linearly increases as we move away from the neutral surface. This means that the resistance to bending provided by any point in the cross section is directly proportional to the distance from the neutral axis squared. We can sum up the resistances to bending then by using the second rectangular area moment of inertia, where our distances are measured from the neutral axis.

$$I=\int_A (dA*d^2) \tag{17.5.1}$$

Assuming we put the origin point at the centroid and that the x-axis is the neutral surface, the distance from the neutral surface to any point is simply the y coordinate of that point. Similarly, if we apply a moment about the y-axis, and had a neutral surface that ran along the y-axis, the distance from the neutral axis would simply be the x coordinate of the point. For this reason, I_{xx} includes y as the distance, while I_{yy} includes x as the distance.

$$I_{xx}=\int_{y_{min}}^{y_{max}} (dA*y^2) \qquad I_{yy}=\int_{x_{min}}^{x_{max}} (dA*x^2) \tag{17.5.2}$$

Calculating the Rectangular Area Moment of Inertia via Integration

To determine the area moment of inertia, start by drawing out the area under analysis, and include the axes you are taking the moment of inertia about. This is important, since the moment of inertia will vary depending on the axis chosen. In cases on bending stresses you will want to put the origin on the neutral surface, which will be at the centroid of the area. If the centroid is not given to you, you will need to determine the centroid as discussed in prior sections.

To take the moment of inertia about the x-axis through this point (I_{xx}) we will use the general formula discussed earlier. We will be moving from bottom to top, integrating the rate of change of the area as we go, and multiplying that by the y-value squared. The rate of change of area (dA) as we move upwards will be the **width** of the object at any given y-value times the rate at which we are moving. Unless the width remains constant, the width will need to be represented as a mathematical function in terms of y.

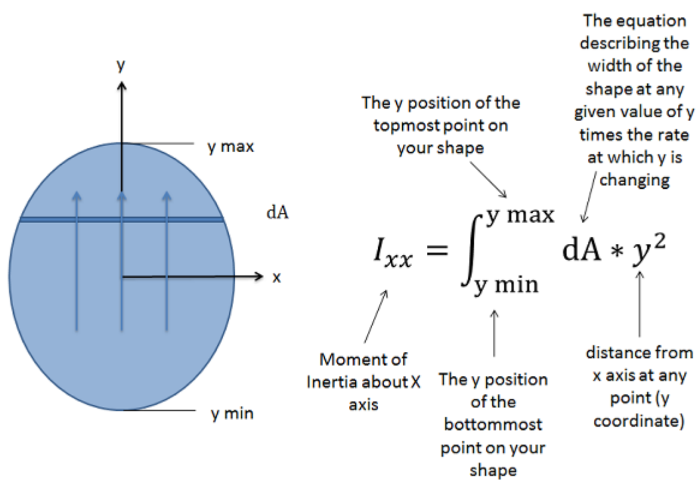


Figure 17.5.3: The rectangular moment of inertia about the x-axis.

To find the moment of inertia about the y-axis through a given point (I_{yy}) we will move left to right, using the distances from the y-axis in our moment integral (in this case the x coordinates of each point). Moving from left to right, the rate of change of the area will be the **height** of the shape at any given x-value times the rate at which we are moving left to right. Again, we will need to describe this with an mathematical function if the height is not constant. We will multiply this function by the x-value squared for the second moment integral, and this will give us the moment of inertia about the y-axis.

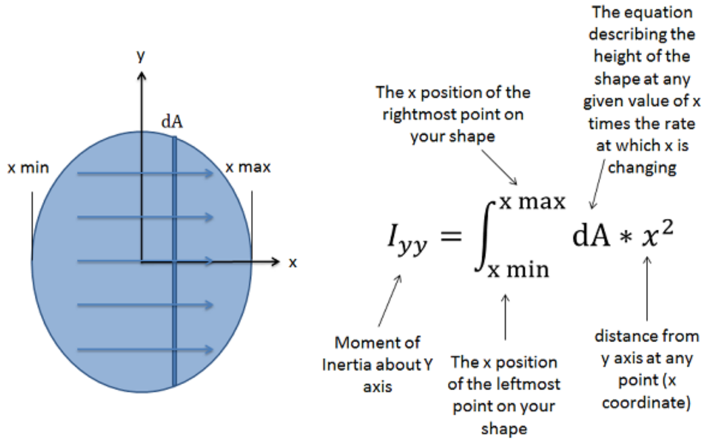


Figure 17.5.4: The rectangular moment of inertia about the y-axis.

Torsional Stresses and the Polar Moment of Inertia

When an object is subjected to a torsional moment, that object will experience internal shearing forces as shown in the diagram below. These stresses are oriented in such a way that they will counteract the torsional moment, but do not exert any net force on the shaft so that shaft stays in equilibrium.

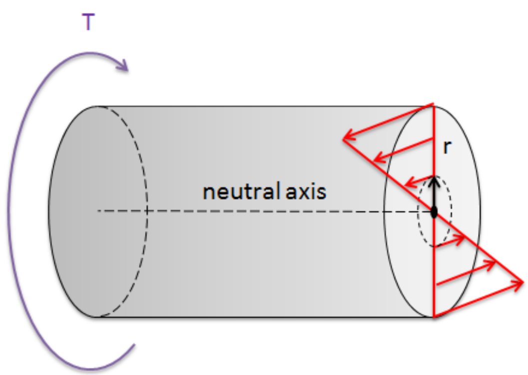


Figure 17.5.5: Torsion causes internal shearing stresses that counteract the moment but exert no net force.

As we can see in the diagram, there is a central axis along which there are no shearing stresses. This is known as the neutral axis, and if there are no other forces present, then this will travel through the centroid of the shaft's cross section. As we move away from the neutral axis in any direction, the stresses will increase linearly. The moment exerted by the stress at any point will be the stress times the moment arm, which also increases linearly as we move away from the neutral axis. This means that the resistance to torsional loading provided by any one point on the cross section is directly proportional the square of the distance between the point and neutral axis. We can sum up the resistances to torsional loading then using the second polar area moment of inertia, where our distances are measured from the neutral axis (r), a single point in the shaft's cross-section.

$$J_{zz} = \int_{r_{min}}^{r_{max}} (dA * r^2) \tag{17.5.3}$$

Calculating the Polar Area Moment of Inertia via Integration

The first step in determining the polar moment of inertia is to draw the area and identify the point about which we are taking the moment of inertia. In the case of torsional loading, we will usually want to pick the point at which the neutral axis travels through the shaft's cross section, which in the absence of other types of loading will be the centroid of the cross section. If the centroid is not clearly identified, you will need to determine the centroid as discussed in previous sections.

To take the moment of inertia about this central point, we will be measuring all distances outward from this point. Rather than moving left to right or top to bottom, we will instead be integrating from the center radiating outward in all directions. We will be going from the minimum distance from the center for our shape (zero unless there is a central hole in our area) to the maximum distance to the center. We will again be integrating the rate of change of area, which in this case will be a function for the **circumference** at a given radius times the rate as which we are moving outwards times the given radius squared.

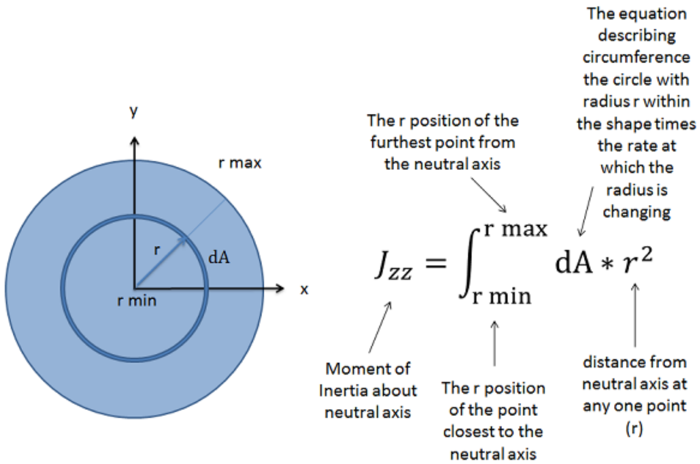


Figure 17.5.6: The polar moment of inertia about the neutral axis.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/gCM6Wd8sa6M.

Example 17.5.1

Find the rectangular moments of inertia for this shape about both the x-axis and y-axis though the centroid. Leave the answer in terms of the generic width (b) and height (h) of the rectangle.

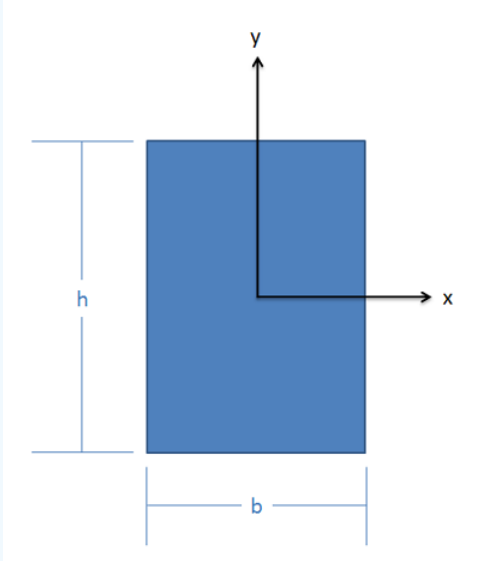


Figure 17.5.7: problem diagram for Example 17.5.1. A solid rectangle is centered at the origin of a Cartesian coordinate plane.

Solution:

Video 17.5.2: Worked solution to example problem 17.5.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/PL6QIBL_rPw.

Example 17.5.2

Find the polar moments of inertia for this circular area about its centroid. Leave the answer in terms of the generic radius R.

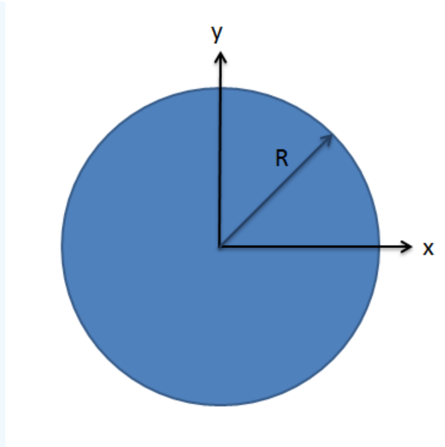


Figure 17.5.8: problem diagram for Example 17.5.2. A solid circle is centered at the origin of a Cartesian coordinate plane.

Solution:

Video 17.5.3: Worked solution to example problem 17.5.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/peqSVmDjThA.

Exercise 17.5.3

Find the polar moment of inertia of this hollow circular shape about its centroid.

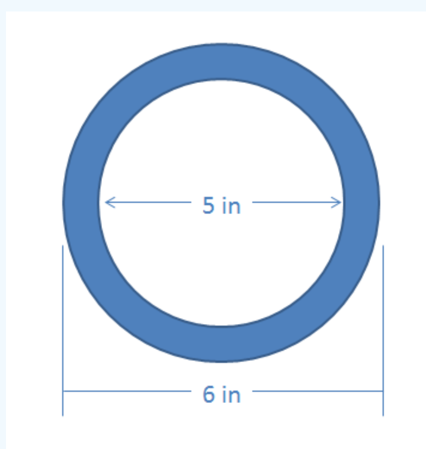


Figure 17.5.9: problem diagram for Example 17.5.3. A disk of diameter 6 inches contains a centered circular cutout of diameter 5 inches.

Solution:

Video 17.5.4: Worked solution to example problem 17.5.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Jw696gGSDM8.

17.6: Mass Moments of Inertia via Integration

The **mass moment of inertia** represents a body's resistance to angular accelerations about an axis, just as mass represents a body's resistance to linear accelerations. This is represented in an equation with the rotational version of Newton's Second Law.

$$F = ma \tag{17.6.1}$$

$$M = I\alpha \tag{17.6.2}$$

Just as with area moments of inertia, the mass moment of inertia can be calculated via moment integrals or via the method of composite parts and the parallel axis theorem. This page will only discuss the integration method, as the method of composite parts is discussed on a separate page.

The Mass Moment of Inertia and Angular Accelerations

The mass moment of inertia is a moment integral, specifically the second polar mass moment integral. To see why this relates moments and angular accelerations, we start by examining a point mass on the end of a massless stick as shown below. Imagine we want to rotate the stick about the left end by applying a moment there. We want to relate the moment exerted to the angular acceleration of the stick about this point.

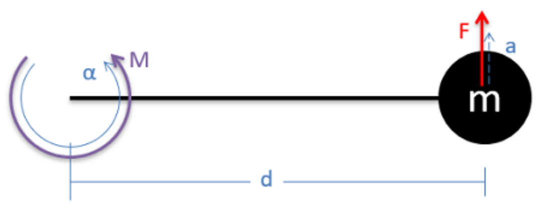


Figure 17.6.1: A point mass on the end of a massless stick. We are attempting to rotate the mass about the stick's left end by exerting a moment there.

To relate the moment and the angular acceleration, we need to start with the traditional form of Newton's Second Law, stating that the force exerted on the point mass by the stick will be equal to the mass times the acceleration of the point mass ($F = m * a$). In this case the moment will be related to the force in that the force exerted on the mass times the length of the stick (d) is equal to the moment. We can also relate the linear acceleration of the mass to its rotational counterpart in that the linear acceleration is the angular acceleration times the length of the rod (d). If we take these two substitutions and put them into the original $F = m * a$ equation, we can wind up with an equation that relates the moment and the angular acceleration for our scenario. A simplified version of this new relationship states that the moment will be equal to the mass times the distance squared times the angular acceleration. This mass-times-distance-squared term (relating the moment and angular acceleration) forms the basis for the mass moment of inertia.

$$F = m * a \tag{17.6.3}$$

$$M = F * d \qquad\qquad a = d * \alpha \tag{17.6.4}$$

$$M = (m * d^2) * \alpha \tag{17.6.5}$$

Taking our situation one step further, if we were to have multiple masses all connected to a central point, the moment and angular acceleration would be related by the sum of all the mass times distance squared terms.

$$M = \sum (m * d^2) * \alpha \tag{17.6.6}$$

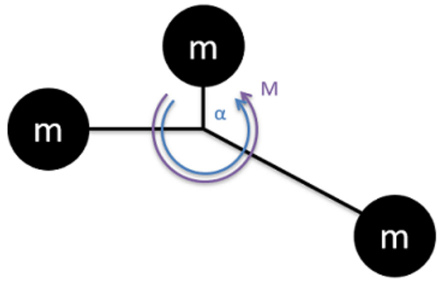


Figure 17.6.2: For systems with multiple masses, we would simply sum up all the mass-times-distance-squared terms to relate moments and angular accelerations.

Taking the final step, rigid bodies with mass distributed over a volume are like an infinite number of small masses about an axis of rotation. Rather than the massless sticks holding everything in place, the mass is simply held in place by the material around it. To relate the moment and angular acceleration in this case, we use integration to add up the infinite number of small mass-times-distance-squared terms.

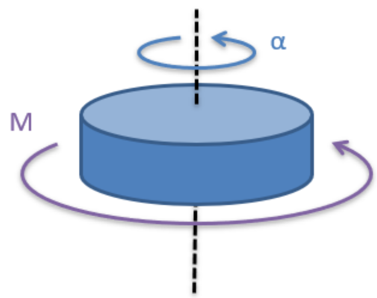


Figure 17.6.3: Approximating a rigid body as an infinite number of infinitely small masses all connected to the axis of rotation, we can sum all the mass-times-distance-squared terms with integration.

$$M=\int_m (dm * d^2) * \alpha \tag{17.6.7}$$

This moment integral which can be calculated for any given shape, called the mass moment of inertia, relates the moment and the angular acceleration for the body about a set axis of rotation.

$$I=\int_m (dm * d^2) \tag{17.6.8}$$

Calculating the Mass Moment of Inertia via Integration

The first step in calculating the mass moment of inertia is to determine the axis of rotation you will be using. Unlike mass, the mass moment of inertia is dependent upon the **point** and **axis** that we are rotating about. We can easily demonstrate this with something like a broomstick, where depending on the position and the direction of the axis we are rotating about, the broomstick can be more or less difficult to rotate.

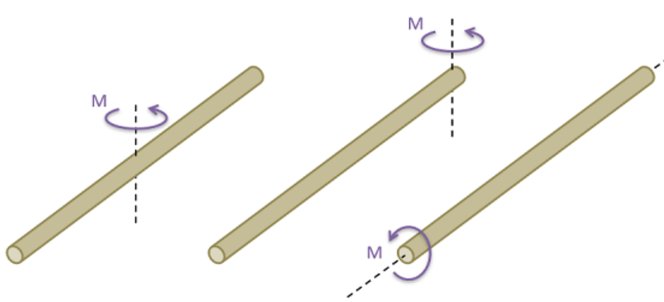


Figure 17.6.4: The mass moment of inertia will vary depending upon the point and direction of the axis of rotation.

After choosing the axis of rotation, it is helpful to draw the shape with the axis of rotation included. This is a polar integral, so we will be taking the mass integral radiating outwards from this axis of rotation.

Also, we are integrating over the mass, and the mass at any given point will be the density times the volume. If the object we are examining has a uniform density, as is often the case, we can pull that density constant outside of the integral, leaving only a integral of the volume. Density is rarely given in these instances, but if you can determine the overall mass and overall volume you can use that as well. If we put all of this into the original equation we had above, we wind up with the following.

$$I=\rho*\int_{r_{min}}^{r_{max}}(dV*r^2)=\frac{m}{V}*\int_{r_{min}}^{r_{max}}(dV*r^2) \tag{17.6.9}$$

For the polar integral, we need to define dV in terms of a radius (r) moving outwards from the axis of rotation. The rate of change of the volume (dV) will be the cylindrical surface area at a given radius times the rate at which that radius is increasing (dr). The height, radius, and holes in this cylindrical surface may all be changing so this dV term may become quite complex, but technically we could find this for mathematical function for any shape. Once we have the dV function in terms of r, we multiply that function by r^2 and we will evaluate the integral.

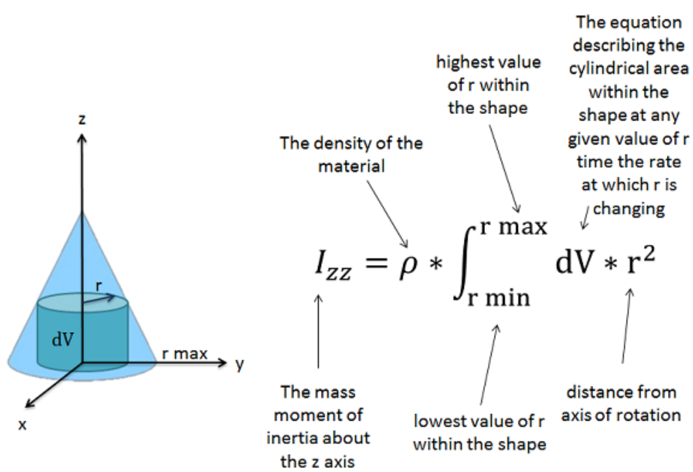


Figure 17.6.5: The mass moment of inertia for a general shape.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/uarIssOmWUU.

Example 17.6.1

A thin circular disk has a mass of 6 kg and a radius of 0.3 meters. Determine the mass moment of inertia for the disk about the z-axis.

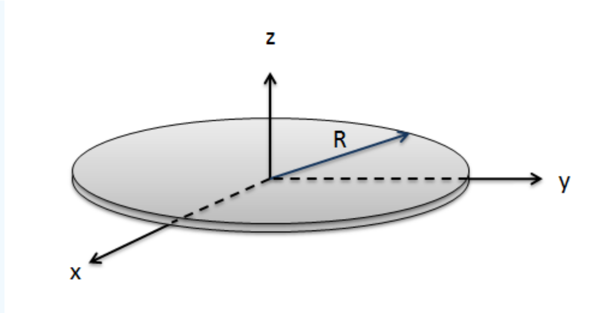


Figure 17.6.6: problem diagram for Example 17.6.1. A thin cylindrical disk lies along the z-axis with its base on the xy-plane, centered at the origin.

Solution:

Video 17.6.2: Worked solution to example problem 17.6.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/e1ZDv6xDUV8.

Example 17.6.2

Determine the mass moment of inertia about the z-axis for this general cone with base radius R, height h, and mass m.

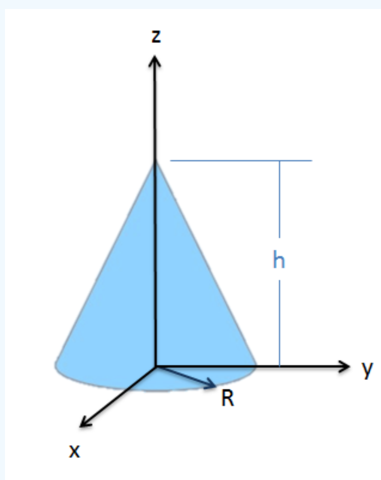


Figure 17.6.7: problem diagram for Example 17.6.2. A cone lies along the positive z-axis, with the base centered at the origin in the xy-plane.

Solution:

Video 17.6.3: Worked solution to example problem 17.6.2, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/rL7xWl9FfWc.

17.7: Moments of Inertia via Composite Parts and Parallel Axis Theorem

As an alternative to integration, both area and mass moments of inertia can be calculated via the method of composite parts, similar to what we did with centroids. In this method we will break down a complex shape into simple parts, look up the moments of inertia for these parts in a table, adjust the moments of inertia for position, and finally add the adjusted values together to find the overall moment of inertia. This method is known as the **method of composite parts**.

A key part to this process that was not present in centroid calculations is the adjustment for position. As discussed on the previous pages, the area and mass moments of inertia are dependent upon the chosen axis of rotation. Moments of inertia for the parts of the body can only be added when they are **taken about the same axis**. However, the moments of inertia in the table are generally listed relative to that shape's centroid. Because each part has its own individual centroid coordinate, we cannot simply add these numbers. We will use something called the **Parallel Axis Theorem** to adjust the moments of inertia so that they are all taken about some standard axis or point. Once the moments of inertia are adjusted with the Parallel Axis Theorem, then we can add them together using the method of composite parts.

The Parallel Axis Theorem

When we calculated the area and mass moments of inertia via integration, one of the first things we had to do was to select a point or axis we were going to take the moment of inertia about. We then measured all distances from that point or axis, where the distances were the moment arms in our moment integrals. Because the centroid of a shape is the geometric center of an area or volume, the average distance from the centroid to any one point in a body is at a minimum. If we pick a different point or axis to take the moment of inertia about, then on average all the distances in our moment integral will be a little bit bigger. Specifically, the further we move from the centroid, the larger the average distances become.

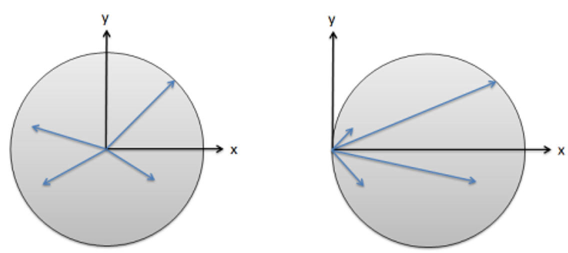


Figure 17.7.1: The distances used in our moment integrals depends on the point or axis chosen. These distances will be at a minimum at the centroid and will get larger as we move further from the centroid.

Though this complicates our analysis, the nice thing is that the change in the moment of inertia is predictable. It will always be at a minimum when we take the moment of inertia about the centroid, or an axis going through the centroid. This minimum, which we will call I_C, is the value we will look up in our moment of inertia table. From this minimum, or unadjusted value, we can find the moment of inertia value about any point I_P by adding an an adjustment factor equal to the area times distance squared for area moments of inertia, or mass times distance squared for mass moments of inertia.

$$I_{xxP} = I_{xxC} + A * r^2 \tag{17.7.1}$$

$$I_{xxP} = I_{xxC} + m * r^2 \tag{17.7.2}$$

This adjustment process with the equations above is the **parallel axis theorem**. The area or mass terms simply represent the area or mass of the part you are looking at, while the distance (r) represents the distance that we are moving the axis about which we are taking the moment of inertia. This may be a vertical distance, a horizontal distance, or a diagonal depending on the axis the moment of inertia is taken about.

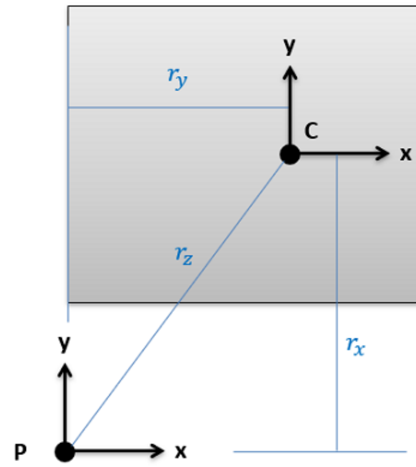


Figure 17.7.2: The distance (r) in the Parallel Axis Theorem represents the distance we are moving the axis we are taking the moment of inertia about.

Say we are trying to find the moments of inertia of the rectangle above about point P. We would start by looking up I_{xx}, I_{yy}, and J_{zz} about the centroid of the rectangle (C) in the moment of inertia table. Then we would add on an area-times-distance-squared term to each to find the adjusted moments of inertia about P. The distance we are moving the x axis for I_{xx} is the vertical distance r_x, the distance we are moving the y-axis for I_{yy} is the horizontal distance r_y, and the distance we would move the z-axis (which is pointing out of the page) for J_{zz} is the diagonal distance r_z.

Center of mass adjustments follow a similar logic, using mass times distance squared, where the distance represents how far you are moving the axis of rotation in three-dimensional space.

Using the Method of Composite Parts to Find the Moment of Inertia

To find the moment of inertia of a body using the method of composite parts, you need to start by breaking your area or volume down into simple shapes. Make sure each individual shape is available in the moment of inertia table, and you can treat holes or cutouts as negative area or mass.

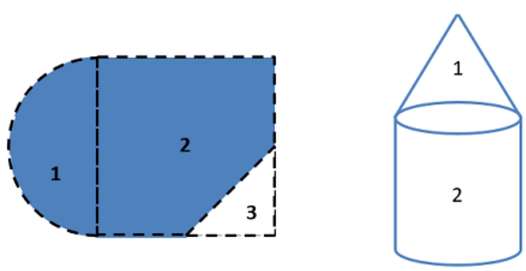
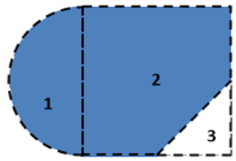


Figure 17.7.3: Start by breaking down your area or volume into simple parts, and number those parts. Holes or cutouts will count as negative areas or masses.

Next you are going to create a table to keep track of values. Devote a row to each part that your numbered earlier, and include a final "total" row that will be used for some values. Most of the work of the method of composite parts is filling in this table. The columns will vary slightly with what you are looking for, but you will generally need the following.



Shape	Area	$\bar{X}$	$\bar{Y}$	$I_{xx\,C}$	$I_{yy\,C}$	r_x	r_y	$I_{xx\,adj}$	$I_{yy\,adj}$
1									
2									
3									
Total									

Figure 17.7.4: Most work in the method of composite parts will revolve around filling out a table such as this one. This table contains the rows and columns necessary to find the rectangular area moments of inertia (I_{xx} and I_{yy}) for this composite body.

- The area or mass for each piece (area for area moments of inertia or mass for mass moments of inertia). Remember that cutouts should be listed as negative areas or masses.
- The centroid or center of mass locations (x, y and possibly z coordinates). Most of the time, we will be finding the moment of inertia about centroid of the composite shape, and if that is not explicitly given to you, you will need to find that before going further. For more details on this, see the page Centroids and Centers of Mass via Method of Composite Parts.
- The moment of inertia values about each shape's centroid. To find these values you will plug numbers for height, radius, mass, etc. into formulas on the moment of inertia table. Do not use these formulas blindly though, as you may need to mentally rotate the body, and thus switch equations, if the orientation of the shape in the table does not match the orientation of the shape in your diagram.
- The adjustment distances (r) for each shape. For this value you will want to determine how far the x-axis, y-axis, or z-axis moves to go from the centroid of the piece to the overall centroid or point you are taking the moment of inertia about. To calculate these values, generally you will be finding the horizontal, vertical, or diagonal distances between piece centroids and the overall centroids that you have listed earlier in the table. See the parallel axis theorem section of this page earlier for more details.
- Finally, you will have a column of the adjusted moments of inertia. Take the original moment of inertia about the centroid, then simply add your area times r^2 term or mass times r^2 term for this adjusted value.

The overall moment of inertia of your composite body is simply the sum of all of the adjusted moments of inertia for the pieces, which will be the sum of the values in the last column (or columns, if you are finding the moments of inertia about more than one axis).

Video 17.7.1: Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/zulGTSWF6xs.

Example 17.7.1

Use the parallel axis theorem to find the mass moment of inertia of this slender rod with mass m and length L about the z-axis at its endpoint.

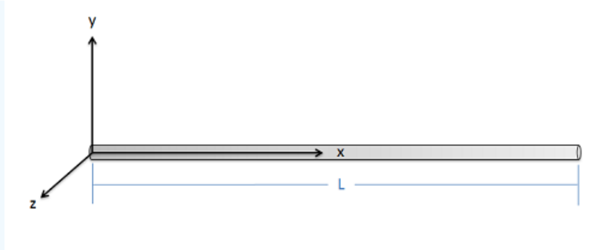


Figure 17.7.5: problem diagram for Example 17.7.1. A rod lies along the positive x-axis of a Cartesian coordinate system, with its left endpoint located at the origin.

Solution:

Video 17.7.2 Worked solution to example problem 17.7.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/4oN0kgDO3Yw.

Example 17.7.2

A beam is made by connecting two 2" x 4" beams in a T-pattern with the cross section as shown below. Determine the location of the centroid of this combined cross section and then find the rectangular area moment of inertia about the x-axis through the centroid point.

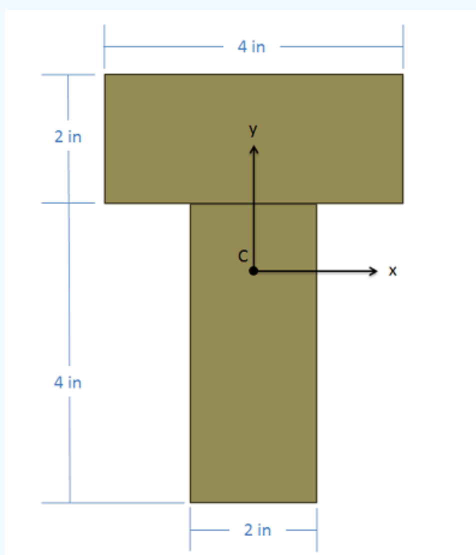


Figure 17.7.6: problem diagram for Example 17.7.2. The top edge of a vertical 2" x 4" beam is centered on and connected to the lower edge of a horizontal 2" x 4" beam, creating a T-shaped assembly.

Solution:

Video 17.7.3 Worked solution to example problem 17.7.2 provided by Dr. Jacob Moore. YouTube source: https://youtu.be/lgXlp2lRaiA.

Example 17.7.3

A dumbbell consists of two spheres of diameter 0.2 meter, each with a mass of 40 kg, attached to the ends of a 0.6-meter-long slender rod of mass 20 kg. Determine the mass moment of inertia of the dumbbell about the y-axis shown in the diagram.

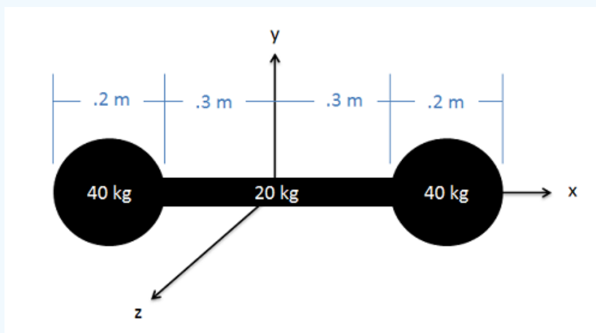


Figure 17.7.7: problem diagram for Example 17.7.3. A dumbbell consists of a slender rod with a sphere attached to each end lying along the x-axis of a Cartesian coordinate system, with its midpoint at the origin.

Solution:

Video 17.7.4 Worked solution to example problem 17.7.3 provided by Dr. Jacob Moore. YouTube source: https://youtu.be/ufewJ7CmvIs.

of the LibreTexts platform; a detailed edit history is available upon request.

17.8: Appendix 2 Homework Problems

Exercise 17.8.1

A shape is bounded on the left by the y-axis, on the bottom by the x-axis, and along its remaining side by the function $y=-\frac{1}{2}x^2+8$. Determine the x and y coordinates of the centroid of this shape via integration. (Hint: for $\bar{y}$, work from the top down to make the math easier.)

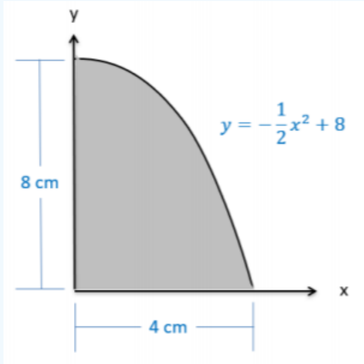


Figure 17.8.1: problem diagram for Exercise 17.8.1. A shape in the first quadrant of a Cartesian coordinate plane is bounded by the intersection of the function $y=-\frac{1}{2}x^2+8$ with the x- and y-axes.

Solution:

$\bar{x}=1.5\ cm,\ \bar{y}=3.2\ cm$

Exercise 17.8.2

Determine the x and y coordinates of the centroid of the shape shown below via integration.

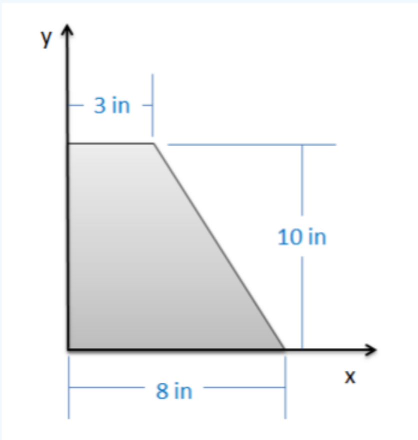


Figure 17.8.2: problem diagram for Exercise 17.8.2. A trapezoid lies in the first quadrant of the Cartesian coordinate plane, with two sides lying along the axes.

Solution:

$\bar{x}=2.94\ in,\ \bar{y}=4.24\ in.$

Exercise 17.8.3

A water tank as shown below takes the form of an inverted, truncated cone. The diameter of the base is 4 ft, the diameter of the top is 8 ft, and the height of the tank is 4 ft. Using integration, determine the height of the center of mass of the filled tank. (Assume the tank is filled with water and the walls have negligible mass.)

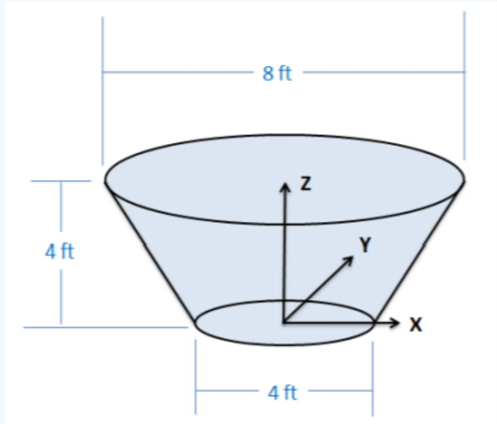


Figure 17.8.3: problem diagram for Exercise 17.8.3. A water tank in the shape of an inverted truncated circular cone, with a wide top and a narrower base.

Solution:

$z_c = 2.43\ ft$

Exercise 17.8.4

Use the method of composite parts to determine the centroid of the shape shown below.

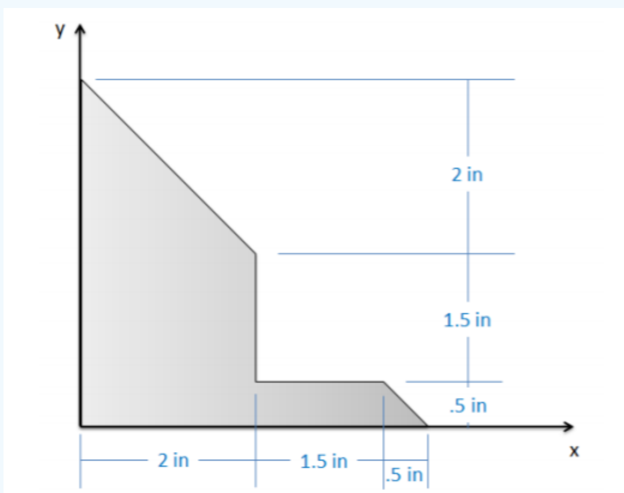


Figure 17.8.4: problem diagram for Exercise 17.8.4. A composite shape in the first quadrant of a Cartesian coordinate plane, with two sides lying along the axes, consists of two trapezoids, or two rectangles and two right triangles.

Solution:

$x_c = 1.14\ in,\ y_c = 1.39\ in$

Exercise 17.8.5

A floating platform consists of a square piece of plywood weighing 50 lbs with a negligible thickness on top of a rectangular prism of a foam material weighing 100 lbs as shown below. Based on this information, what is the location of the center of mass for the floating platform?

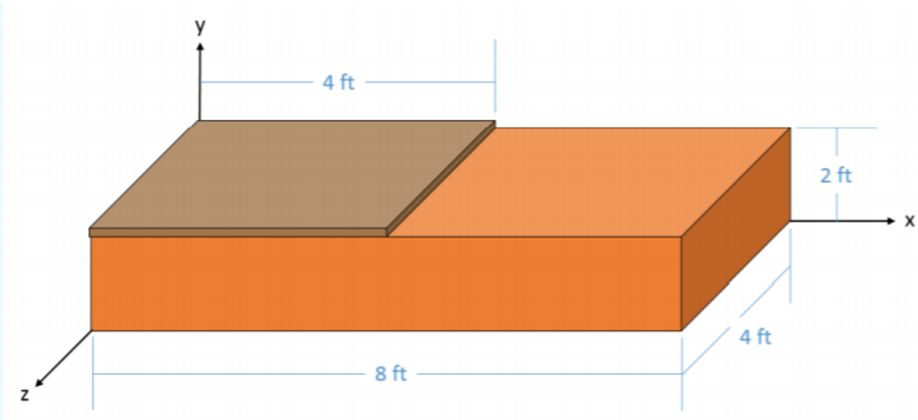


Figure 17.8.5: problem diagram for Exercise 17.8.5. A foam rectangular prism with one half of its top face covered by a plywood square lies in the first octant of a Cartesian coordinate system.

Solution:

$x_c = 3.33\ ft,\ y_c = 1.33\ ft,\ z_c = 2\ ft$

Exercise 17.8.6

Use the integration method to find the moments of inertia for the shape shown below…

- About the x-axis through the centroid.
- About the y-axis through the centroid.

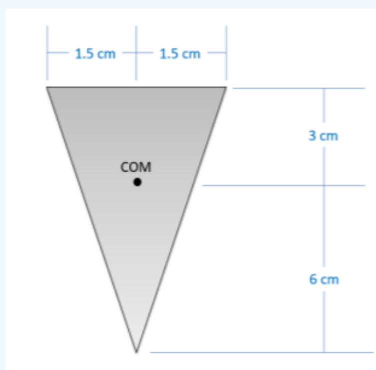


Figure 17.8.6: problem diagram for Exercise 17.8.6. An downwards-facing isosceles triangle has a horizontal base and a center of mass on its line of symmetry.

Solution:

$I_{xx} = 6.075 * 10^{-7}\ m^4$

$I_{yy} = 5.0625 * 10^{-8}\ m^4$

Exercise 17.8.7

Use the integration method to find the polar moment of inertia for the semicircle shown below about point O.

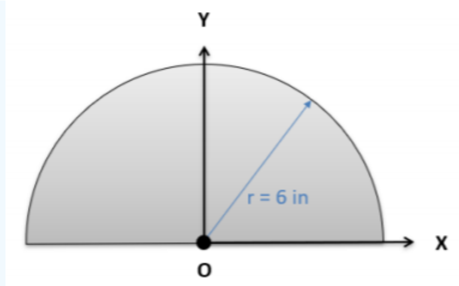


Figure 17.8.7: problem diagram for Exercise 17.8.7. A semicircle lies with its straight edge centered on the x-axis of a Cartesian coordinate plane, with origin O.

Solution:

$J_{zz} = 1017.9\ in^4$

Exercise 17.8.8

A plastic beam has a square cross-section with semicircular cutouts on the top and bottom as shown below. What is the area moment of inertia of the beam's cross section about the x and y axes through the center point?

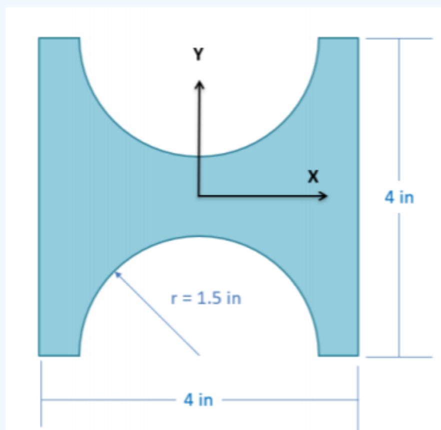


Figure 17.8.8: problem diagram for Exercise 17.8.8. A beam cross-section consists of a square with semicircular cutouts centered on the top and bottom sides.

Solution:

$I_{xx} = 7.08\ in^4$, $I_{yy} = 17.36\ in^4$

Exercise 17.8.9

A piece of angled steel has a cross section that is 1 cm thick and has a length of 6 cm on each side as shown below. What are the x and y area moments of inertia through the centroid of the cross section?

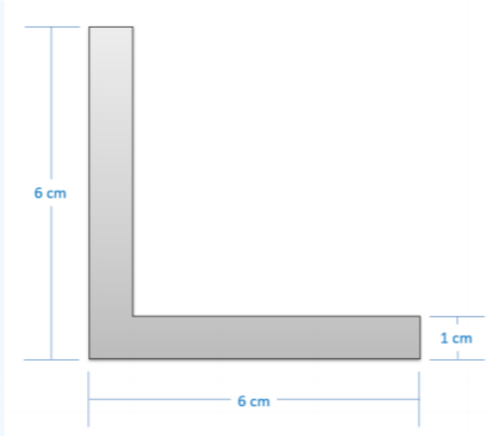


Figure 17.8.9: problem diagram for Exercise 17.8.9. A part cross-section consists of an L shape composed of two identically sized rectangles.

Solution:

$$I_{xx} = I_{yy} = 35.462\, cm^4 = 3.546 * 10^{-7}\, m^4$$

Exercise 17.8.10

The pendulum in an antique clock consists of a brass disc with a mass of 0.25 kg and diameter of 6 cm at the end of a slender wooden rod with a mass of 0.1 kg. Determine the mass moment of inertia of the pendulum about the top of the rod.

Figure 17.8.10: problem diagram for Exercise 17.8.10. A clock pendulum is represented as a vertical wooden rod with a brass disc attached to its bottom edge.

Solution:

$$I_{zz} = 0.02026\, kg\, m^2$$

Exercise 17.8.11

A space telescope can be approximated as a 600-kg cylinder with a 4-meter diameter and 4-meter height attached to two 100-kg solar panels as shown below. What is the approximate mass moment of inertia for the space telescope about the y-axis shown?

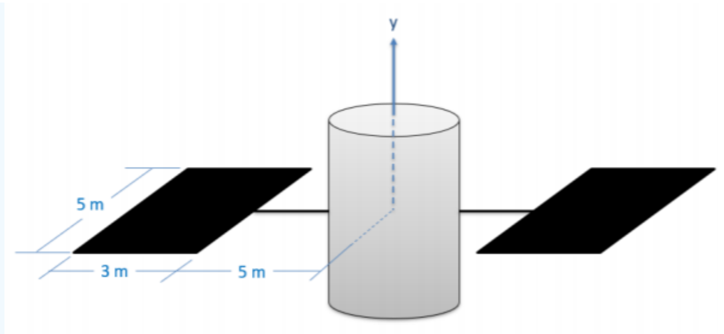


Figure 17.8.11: problem diagram for Exercise 17.8.11. A space telescope is represented as a vertical cylinder with two horizontal rods extending from midway along the cylinder's height, each supporting a solar panel.

Solution:

$$I_{yy} = 10,216.7\ kg\, m^2$$

Exercise 17.8.12

A flywheel has an original weight of 15 pounds and a diameter of 6 inches. To reduce the weight, four two-inch diameter holes are drilled into the flywheel, each leaving half an inch to the outside edge as shown below. What was the original polar mass moment of about the center point? Assuming a uniform thickness, what is the new mass moment of inertia after drilling in the holes? (Hint: holes count as negative mass in the mass moment calculations.)

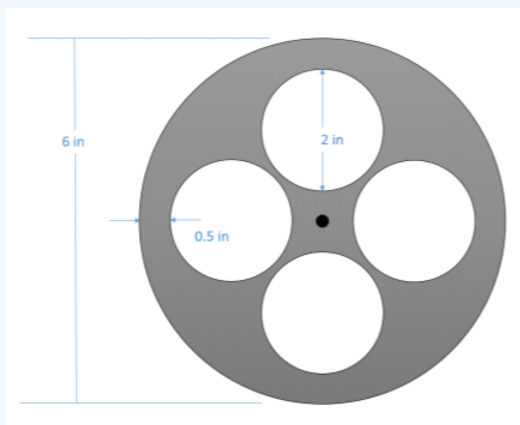


Figure 17.8.12: problem diagram for Exercise 17.8.12. A flywheel consists of a circular disk with four circular holes drilled in a radially symmetric pattern about its center point.

Solution:

$$J_{withoutholes} = 0.01456\ slug\, ft^2$$

$$J_{withholes} = 0.01060\ slug\, ft^2$$

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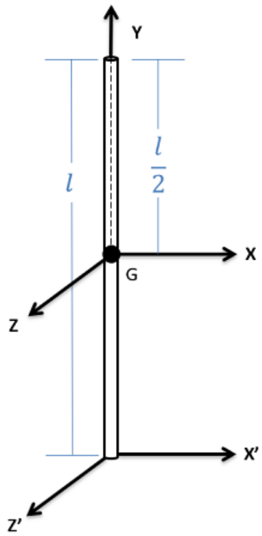
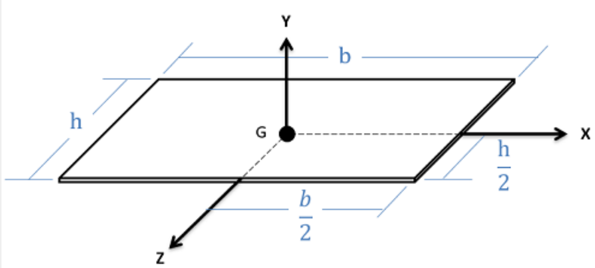
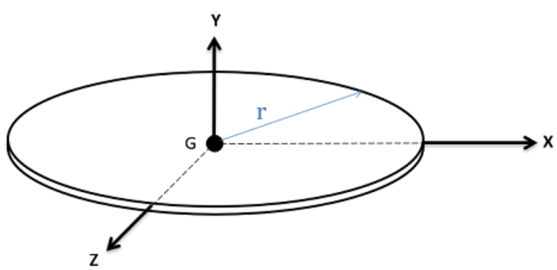
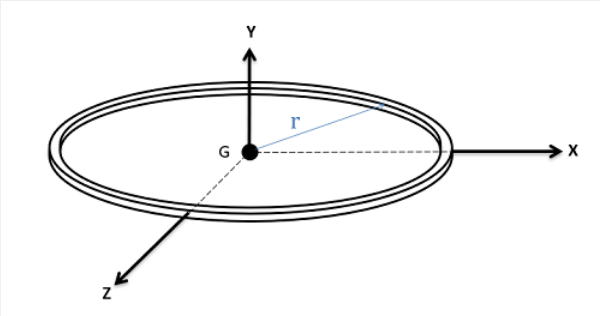
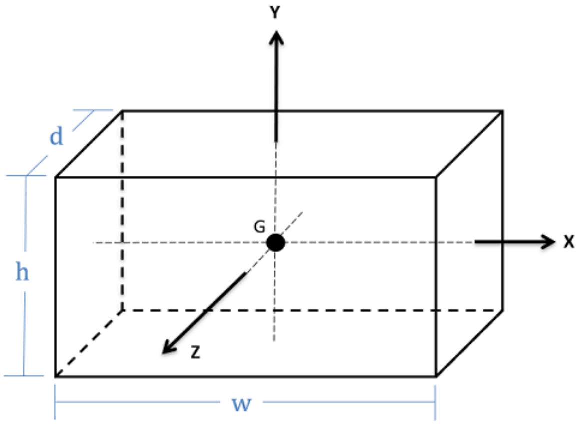
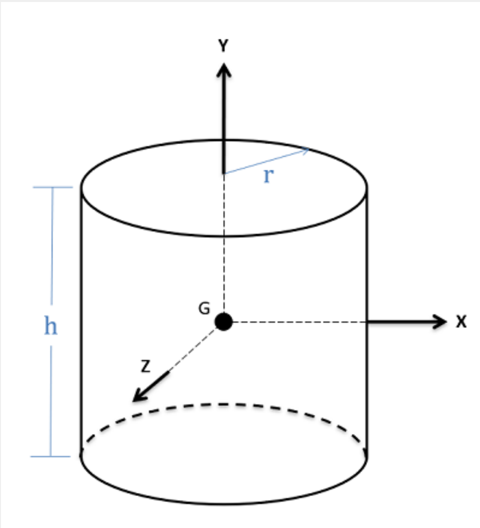
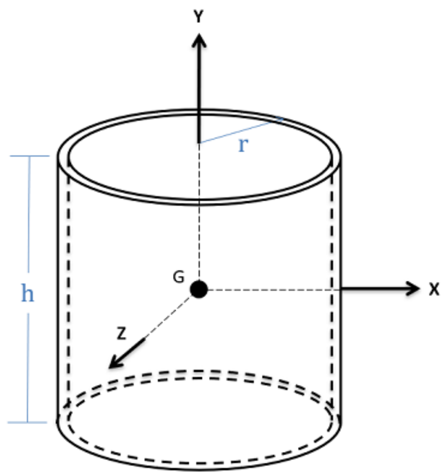
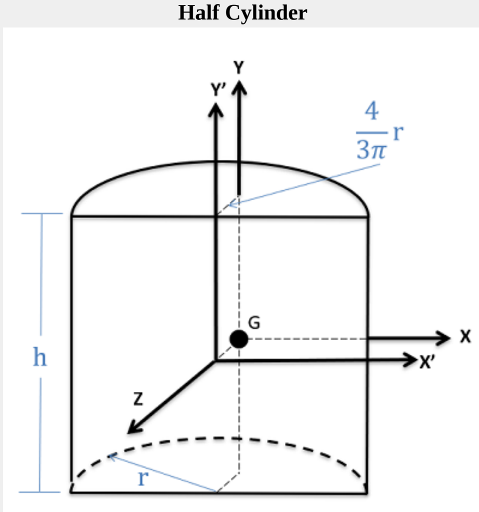
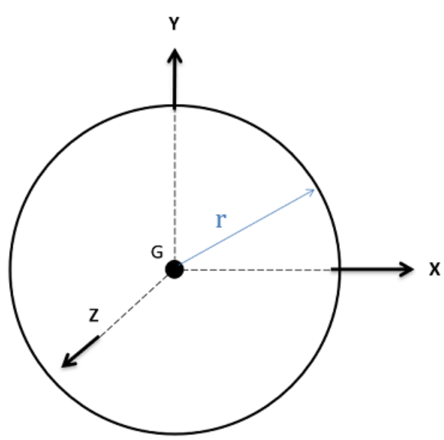
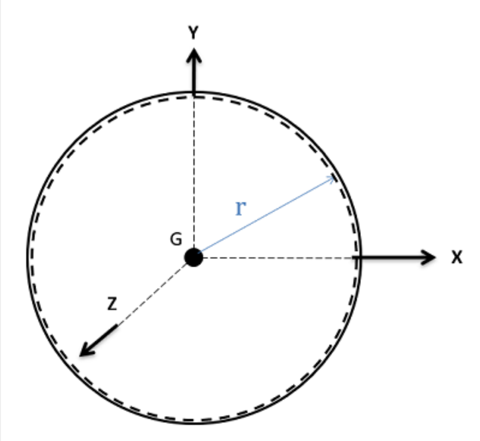
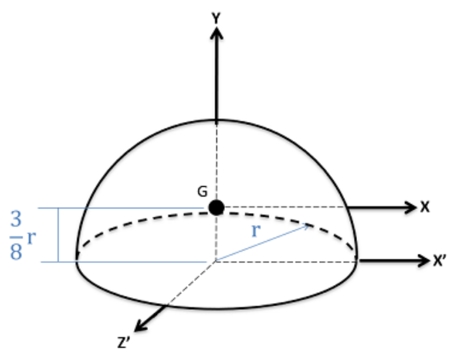
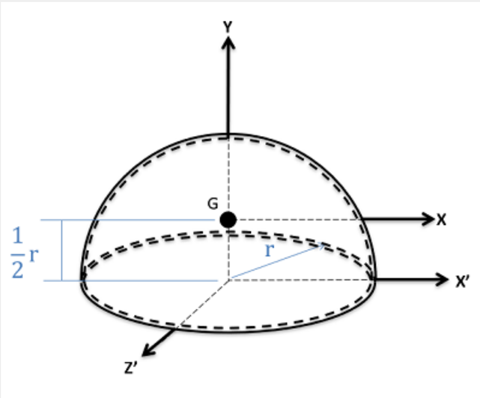
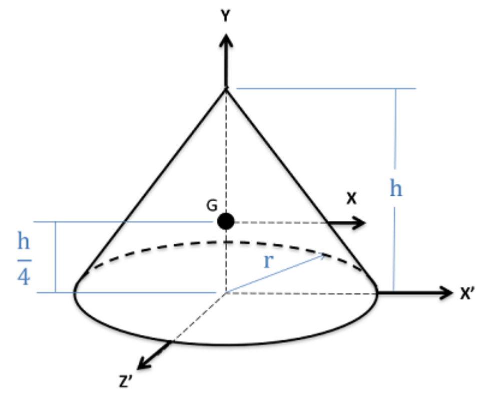
Center of Mass and Mass Moments of Inertia for Homogeneous 3D Bodies

Shape with Volume and Center of Mass Location Shown	Mass Moments of Inertia
Slender Rod 	$I_{xx} = I_{zz} = \frac{1}{12}ml^2$ $I_{yy} = 0$ $I_{x'x'} = I_{z'z'} = \frac{1}{3}ml^2$
Flat Rectangular Plate 	$I_{xx} = \frac{1}{12}mh^2$ $I_{yy} = \frac{1}{12}m(h^2 + b^2)$ $I_{zz} = \frac{1}{12}mb^2$
Flat Circular Plate 	$I_{xx} = I_{zz} = \frac{1}{4}mr^2$ $I_{yy} = \frac{1}{2}mr^2$

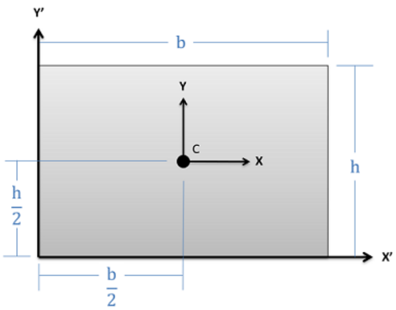
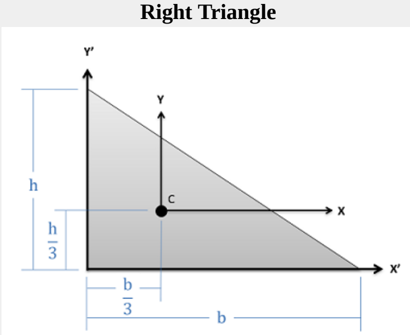
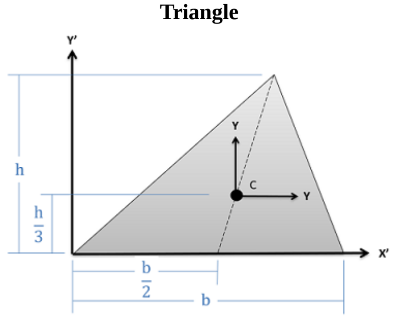
Shape with Volume and Center of Mass Location Shown	Mass Moments of Inertia
Thin Circular Ring 	$I_{xx} = I_{zz} = \frac{1}{2} m r^2$ $I_{yy} = m r^2$
Rectangular Prism  $Volume = dwh$	$I_{xx} = \frac{1}{12} m(h^2 + d^2)$ $I_{yy} = \frac{1}{12} m(d^2 + w^2)$ $I_{zz} = \frac{1}{12} m(h^2 + w^2)$
Cylinder  $Volume = \pi r^2 h$	$I_{xx} = I_{zz} = \frac{1}{12} m(3r^2 + h^2)$ $I_{yy} = \frac{1}{2} m r^2$

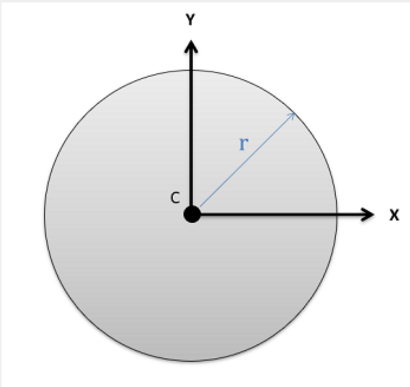
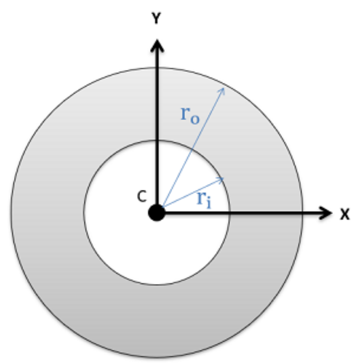
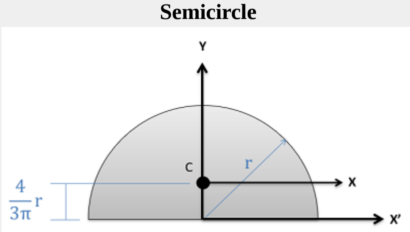
Shape with Volume and Center of Mass Location Shown	Mass Moments of Inertia
Thin Cylindrical Shell  	$I_{xx} = I_{zz} = \frac{1}{6}m(3r^2 + h^2)$ $I_{yy} = mr^2$
Half Cylinder  $Volume = \frac{1}{2}\pi r^2 h$	$I_{xx} = I_{zz} = \left(\frac{1}{4} - \frac{16}{9\pi^2}\right)mr^2 + \frac{1}{12}mh^2$ $I_{yy} = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)mr^2$ $I_{xx'} = I_{zz'} = \frac{1}{12}m(3r^2 + h^2)$ $I_{yy'} = \frac{1}{2}mr^2$
Sphere  $Volume = \frac{4}{3}\pi r^3$	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$

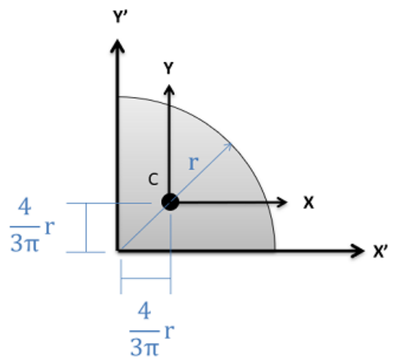
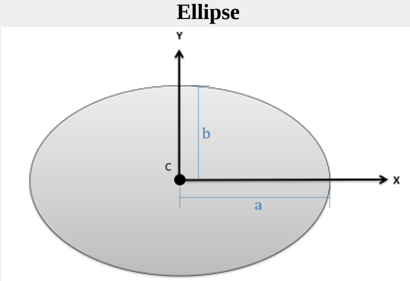
Shape with Volume and Center of Mass Location Shown	Mass Moments of Inertia
Spherical Shell 	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{3}mr^2$
Hemisphere  $Volume = \frac{2}{3}\pi r^3$	$I_{xx} = I_{zz} = \frac{83}{320}mr^2$ $I_{yy} = \frac{2}{5}mr^2$ $I_{xx'} = I_{zz'} = \frac{2}{5}mr^2$
Hemispherical Shell 	$I_{xx} = I_{zz} = \frac{5}{12}mr^2$ $I_{yy} = \frac{2}{3}mr^2$ $I_{xx'} = I_{zz'} = \frac{2}{3}mr^2$

Shape with Volume and Center of Mass Location Shown	Mass Moments of Inertia
Right Circular Cone  $Volume = \frac{1}{3}\pi r^2 h$	$I_{xx} = I_{zz} = \frac{3}{80}m(4r^2 + h^2)$ $I_{yy} = \frac{3}{10}mr^2$ $I_{xx'} = I_{zz'} = \frac{1}{20}m(3r^2 + 2h^2)$

Centroids and Area Moments of Inertia for 2D Shapes

Shape with Area and Centroid Location Shown	Rectangular Area Moments of Inertia	Polar Area Moments of Inertia
Rectangle  $Area = bh$	$I_x = \frac{1}{12}bh^3$ $I_y = \frac{1}{12}b^3h$	$J_z = \frac{1}{12}bh(b^2+h^2)$
Right Triangle  $Area = \frac{1}{2}bh$	$I_x = \frac{1}{36}bh^3$ $I_y = \frac{1}{36}b^3h$ $I_{x'} = \frac{1}{12}bh^3$ $I_{y'} = \frac{1}{12}b^3h$	
Triangle  $Area = \frac{1}{2}bh$	$I_x = \frac{1}{36}bh^3$ $I_{x'} = \frac{1}{12}bh^3$	

Shape with Area and Centroid Location Shown	Rectangular Area Moments of Inertia	Polar Area Moments of Inertia
Circle  $Area = \pi r^2$	$I_x = \frac{\pi}{4} r^4$ $I_y = \frac{\pi}{4} r^4$	$J_z = \frac{\pi}{2} r^4$
Circular Annulus  $Area = \pi(r_o^2 - r_i^2)$	$I_x = \frac{\pi}{4}(r_o^4 - r_i^4)$ $I_y = \frac{\pi}{4}(r_o^4 - r_i^4)$	$J_z = \frac{\pi}{2}(r_o^4 - r_i^4)$
Semicircle  $Area = \frac{\pi}{2} r^2$	$I_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$ $I_y = \frac{\pi}{8} r^4$ $I_{x'} = \frac{\pi}{8} r^4$	$J_z = \left(\frac{\pi}{4} - \frac{8}{9\pi}\right) r^4$

Shape with Area and Centroid Location Shown	Rectangular Area Moments of Inertia	Polar Area Moments of Inertia
Quarter Circle  $Area = \frac{\pi}{4}r^2$	$I_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)r^4$ $I_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)r^4$ $I_{x'} = \frac{\pi}{16}r^4$ $I_{y'} = \frac{\pi}{16}r^4$	$J_z = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4$
Ellipse  $Area = \pi ab$	$I_x = \frac{\pi}{4}ab^3$ $I_y = \frac{\pi}{4}a^3b$	